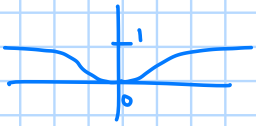
4th march:

Recall: In tut 3.3 $f:\mathbb{R}\to\mathbb{R}$

$$f(x) = \begin{cases} e^{-1/x^2} ; & x\neq 0 \\ 0 ; & x=0\end{cases}$$

is a C^∞ function

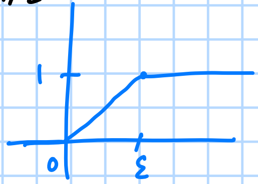


(proved before)

In tut 4.2: $\exists C^\infty$ $f:\mathbb{R}\to\mathbb{R}$ which is positive on $(-1,1)$ and 0 elsewhere

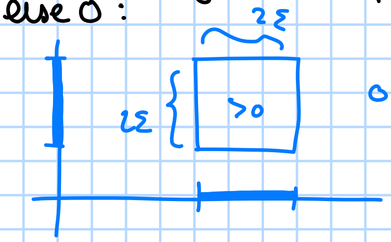
(some combination of e^{-1/x^2})

In tut 5.5: $\exists C^\infty$ function $g:\mathbb{R}\to[0,1]$ s.t $g(x)=0$ for $x\leq 0$ and $g(x)=1$ for $x\geq\varepsilon$

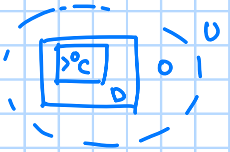


$$\left(g(x) = \frac{\int_0^x f(x)dx}{\int_0^\varepsilon f(x)dx} \right)$$

In tut 6.6: $\exists C^\infty$ function g which is positive on $(a^1-\varepsilon, a^1+\varepsilon)\times\dots\times(a^n-\varepsilon,a^n+\varepsilon)$ else 0:



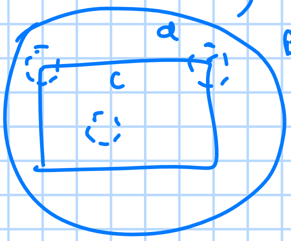
proposition: let $U\subseteq\mathbb{R}^n$ be open, let $C\subset U$ be compact. Then, $\exists$ a closed set D s.t $C\subseteq \text{int}(D)\subset D\subset U$, and there exist a non-negative C^∞ function $f:U\to\mathbb{R}$ s.t $f(x)>0$ for $x\in C$ and $f(x)=0$ $\forall x\in U\setminus D$



proof: C is compact, $A=\mathbb{R}^n\setminus U$ is closed,

so $\exists d>0$ s.t

$|y-x|>d$ $\forall y\in A, x\in C$ (from week 1)

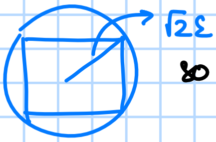


$B_d(x)\subseteq U$ $\forall x\in C$

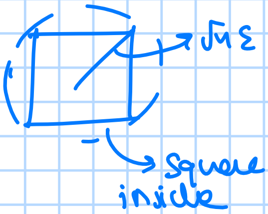
$\Rightarrow \forall x\in C$, $B_d(x)\subseteq U$

then for $\varepsilon>0$ s.t $\sqrt{n}\varepsilon<d$

$\Rightarrow D_{\sqrt{n}\varepsilon}(x)\subset D_d(x)\subseteq U$



so $D_{\sqrt{n}\varepsilon}\subseteq \overline{D_{\sqrt{n}\varepsilon}(x)}\subseteq B_d(x)\subseteq U$



then $D_\varepsilon(x)=(x^1-\varepsilon,x^1+\varepsilon)\times\dots\times(x^n-\varepsilon,x^n+\varepsilon)$

or $D_\varepsilon(x)\subseteq D_{\sqrt{n}\varepsilon}(x)$

so $\{D_\varepsilon(x) \mid x\in C\}$ is an open cover for C & C is compact

$\Rightarrow \exists \{D_\varepsilon(x_1), D_\varepsilon(x_2),\dots D_\varepsilon(x_r)\}$ subcover of C

and $D=\overline{D_\varepsilon(x_1)}\cup\dots\cup\overline{D_\varepsilon(x_k)}\subseteq U$

closed

let g_{x_K} be C^∞ s.t $g_{x_K} > 0$ on $D_\varepsilon(x_K) > 0$ and 0 elsewhere.

so $f(x) = g_{x_1}(x) + g_{x_2}(x) + \cdots g_{x_r}(x)$

then $f \geq 0$; f is C^∞, and $\forall x \in C \Rightarrow x \in D_\varepsilon(x_i)$ for some i

$$\Rightarrow f(x) \geq g(x_i) > 0$$

$$\Rightarrow f(x) > 0 \ \forall x \in C$$ (this is by construction)

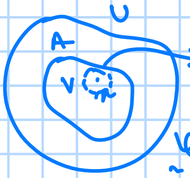
and, if $x \in U \setminus D$, then $\forall i$, $x \notin D_\varepsilon(x_i)$, so

$$g_{x_i}(x) = 0 \ \forall i$$
$$\Rightarrow f(x) = 0$$

Defn: let $A \subset \mathbb{R}^n$, $\mathcal{O}$ is an open cover of A, $U \subseteq \mathbb{R}^n$ open set s.t $A \subseteq U$

Defn: A C^∞ partition of unity for set A is a collection Φ of C^∞ functions

$\varphi : U \to [0,1]$ s.t ($\varphi \in \Phi$)



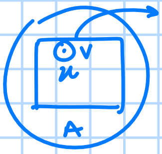
almost all 0
$\varphi_1, \varphi_2 ..$
~ φ_n non zero

① $\forall x \in A \Rightarrow \exists$ open set $V \ni x$ s.t all but finitely many φ are 0 on V.

② for each $x \in A$, we have $\sum_{\varphi \in \Phi} \varphi(x) = 1$ ↳ Addition of all $=1$
φ has $[0,1]$ as range

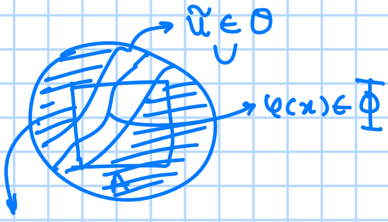
①, ② gives C^∞ partition of unity at A

③ if Φ also satisfies $\forall \varphi \in \Phi$, $\exists$ open $\tilde{U} \in \mathcal{O}$ s.t $\varphi = 0$ outside of some closed set cont $\tilde{U}$.
we say Φ is C^∞ partition of unity for A, subordinate to cover $\mathcal{O}$.



on V: $\sum_{\varphi \in \Phi} \varphi(x) = 1$
U
and only finitely many φ non-zero

($\forall$ function $\exists \tilde{V} \in \mathcal{O}$ s.t $\varphi = 0$ outside some closed set cont $\tilde{V}$)

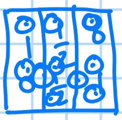


$\tilde{U} \in \mathcal{O}$
U

$\varphi(x) \in \Phi$

0 outside some closed set containing $\tilde{U}$

Exe: partition of unity (not C^∞)



$\varphi_1 : [0,1] \times [0,1] \to [0,1]$

$$\varphi_1 = \begin{cases} 1 & ; \ 0 \leq x \leq 1/3 \\ 1/2 & ; \ 1/3 < x_1 < 2/3 \\ 0 & ; \ \text{otherwise} \end{cases}$$

$\sum \varphi(x) = 1$
$\forall x \in [0,1] \times [0,1]$

$$\varphi_2 = \begin{cases} 1 & ; \ \text{otherw} \\ 1/2 & ; \ \text{others} \\ 0 & ; \ \text{others} \end{cases}$$

Theorm: let $A \subset \mathbb{R}^n$ and $\mathcal{O}$ be an open cover of A, then $\exists$ a C^∞ partition of unity Φ for A, subordinate to the cover $\mathcal{O}$. (this is that $\exists$ such Φ $\forall A \subseteq \mathbb{R}^n$)

proof:

Case I: A is compact set

(specific case) $\Rightarrow$ A can be covered by finitely many sets $U_1, \ldots\ldots U_n$ in $\mathcal{O}$.

and it is enough to construct a partition of unity for A which is subordinate to cover $\{U_1, \ldots\ldots U_n\}$ (3rd point $\tilde{U} = U_i$ in definition)

firstly we will find compact set $D_i \subset U_i$ whose interior cover A

wlog $\forall i$, $\{U_1 \ldots U_n\} \setminus \{U_i\}$ is not a cover of A

let $C_1 = A \setminus (U_2 \cup \ldots U_n)$

is compact

$\Rightarrow \underset{\text{copact}}{\underset{\downarrow}{C_1}} \subset \underset{\text{open}}{\underset{\downarrow}{U_1}}$ and C_1 is compact

then $\exists$ compact D_1 s.t

$C_1 \subset \text{int}(D_1) \subset D_1 \subset U_1$

$\Rightarrow \{D_1, U_2, \ldots U_n\}$ covers A

moreover:

$\{D_1, D_2, \ldots\ldots D_n\}$ covers A (By induction)

i.e $\{D_1 \ldots D_k, U_{k+1}, \ldots U_n\}$ covers A

then

$C_{k+1} = A \setminus (\text{int} D_1 \cup \text{int} D_2 \ldots \text{int} D_k \cup U_{k+2} \ldots U_n)$

then $C_{k+1} \subset U_{k+1}$

& C_{k+1} is compact

$\Rightarrow \exists$ compact D_{k+1} s.t

$C_{k+1} \subset \text{int}(D_{k+1}) \subset D_{k+1} \subset U_{k+1}$

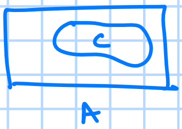
$\Rightarrow \{D_1, \ldots D_{k+1}, U_{k+2}, \ldots U_n\}$ covers A

so by induction we have constructed compact sets $\{D_1, \ldots D_n\}$ s.t $D_i \subset U_i$ and int D_i cover A

$\{\text{int}(D_1), \ldots \text{int}(D_n)\}$ covers A

6th march:

Recap: we defined integration over rectangle, then we did χ_C by covering C to A



(so $\forall C \subseteq V$, C is compact, $\exists D$ ← closed s.t $C \subseteq \text{int}(D) \subset D \subseteq U$
s.t $\exists f : V \longrightarrow \mathbb{R}$ which $f(x) > 0 \;\forall x \in C$
$f(x) = 0 \;\forall x \in U \setminus D$)

let $A \subset \mathbb{R}^n$, and $\mathcal{O}$ be an open cover for A. let U_1 be an open set s.t $A \subset U_1$

Defn: (C^∞ partition of unity for A) is called a collection Φ of C^∞ functions $\varphi : U_1 \longrightarrow [0,1]$ satisfying the following:

① $\forall x \in A$, $\exists$ open $V \ni x$ s.t all but finitely many φ are 0 on V.

② $\forall x \in A$, $\sum_{\varphi \in \Phi} \varphi(x) = 1$

Defn: (C^∞ partition of unity for A subordinate to $\mathcal{O}$) is when

③ $\forall \varphi \in \Phi$, $\exists \tilde{U} \in \mathcal{O}$ ($\tilde{U}$ is open) s.t $\varphi = 0$ outside of some closed set contained in $\tilde{U}$.

Theorem: let $A \subset \mathbb{R}^n$ and $\mathcal{O}$ be an open cover of A, then $\exists$ a C^∞ partition of unity Φ for A, subordinate to the cover $\mathcal{O}$.

(A can be any set and $\mathcal{O}$ any open cover)

proof:

Case I: A is compact then

$\overset{\curvearrowleft}{\text{subcover of } A}$ $\bar{\mathcal{O}} = \{U_1, U_2 \ldots U_k\}$ $\quad U_i \in \mathcal{O} \;\forall i = 1, 2 \ldots k$

and now

$\exists$ $\{\text{int } D_1, \text{int } D_2, \ldots \text{int } D_k\}$ cover by induction

s.t $\quad \text{int } D_i \subset \bar{D_i} \subset U_i$

$\bar{D_i} = D_i \rightarrow D_i$ are closed from condition

now, we will use the proposition, let ψ_i be a non-negative C^∞ function s.t it is > 0 on D_i and 0 on $U_i \setminus \tilde{D_i}$ for some closed set $\tilde{D_i}$ satisfying $D_i \subset \text{int}(\tilde{D_i})$ and $\tilde{D_i} \subset U$.

(By proposition)

($\because \forall D_i$ compact, $\tilde{D_i}$ is a closed set $\tilde{D_i} \subseteq U_i$
$\exists \psi_i : U_i \longrightarrow \mathbb{R}$
s.t $\psi_i > 0 \;\forall x \in \text{int } \tilde{D_i}$ & $\psi_i = 0 \;\forall x \in U_i \setminus \tilde{D_i}$)

since $\{D_1, D_2 \ldots D_k\}$ covers A we have

$\psi_1(x) + \cdots + \psi_k(x) > 0 \;\forall x \in W$ where $W = \bigcup_{i=1}^{k} \text{int}(\tilde{D_i})$

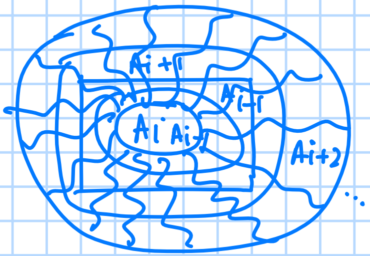
$A \subset W$ ← cover of A

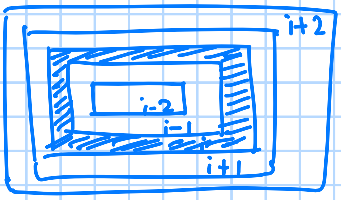
now, on W, define

$$\tilde{\varphi_i}(x) := \frac{\psi_i(x)}{\psi_1(x) + \psi_2(x) + \cdots \psi_k(x)}$$

let $f : W \longrightarrow [0,1]$ be a C^∞ function which is 1 on A, 0 outside some closed set in W (form least possible) then $\Phi = \{f \cdot \tilde{\varphi_1}, f \cdot \tilde{\varphi_2}, \ldots f \cdot \tilde{\varphi_k}\}$ is the desired partition of unity

case II : $A = A_1 \cup A_2 \cup \cdots$
where each A_i is compact
and $A_i \subset \text{int}(A_{i+1})$





for each i, let $\mathcal{O}_i := \{ u \cap \text{interior}(A_{i+2} \setminus A_{i-2}) : u \in \mathcal{O} \}$

$\mathcal{O}_i$ is the open cover for $B_i = A_i \setminus \text{int}(A_{i-1})$
B_i is compact

then by case I, $\exists\ \Phi_i$ for B_i subordainate to cover $\mathcal{O}_i$

so, $\forall\ x \in A$, the sum
$$\sigma(x) = \sum_{\substack{\varphi \in \Phi_i \\ \forall i}} \varphi(x)$$
is finite in some open set cont x.

($\because$ if $x \in A_i$ then $\varphi(x) = 0$
$\forall\ \varphi \in \Phi_j$ for $j \geq 2+i$
as each $\omega \in \mathcal{O}_i$ satisfies
$\omega \subseteq \text{int}(A_{i+2} \setminus A_{i-2})$)

now let $\varphi \in \Phi_i\ \forall i \Rightarrow \tilde{\varphi}(x) = \dfrac{\varphi(x)}{\sigma(x)}$

then set of all $\tilde{\varphi}$ is the required partion of unity.

case III : A is open, In this case let $A_i := \{ x \in A \mid |x| \leq i$ and distance of x to boundry (A) is $\geq \frac{1}{i} \}$

this set is closed and bounded $\Rightarrow$ compact

and $A_i \subseteq \text{int}(A_{i+1})$
so $A = A_1 \cup A_2 \cup \cdots$
$\Rightarrow$ By case II
this also exist

case IV : A is arbitrary, then let $B := \bigcup_{u \in \mathcal{O}} u$ (open)

by case III, $\exists$ a partion of unity Φ for B

$A \subseteq B \Rightarrow \Phi$ is also a patition of unity for A.

10th March:

Recap: Properties of integral
Partitions of unity → defined
→ existance

Theorm: let A be a rectangle in $\mathbb{R}^n$. let $f: A \to \mathbb{R}$ be integrable on A.
(a) If f vanishes except on a set of measure 0, then $\int_A f = 0$

(b) If $f \geq 0$ and if $\int_A f = 0$ then f vanishes except on a set of measure 0.

proof: (a) suppose $\int f$ vanishes exept on a set E of measure 0. let P be a partion of A.
If S is (non-degerate) subrectangle of the partion P, then S is not contained in E.
(non-degerate rectagles do not have measure 0)

∴ $f = 0$ on some point in S.
$\Rightarrow m_S(f) \leq 0$
& $M_S(f) \geq 0$
$\Rightarrow L(f,P) \leq 0$
$U(f,P) \geq 0$

(Partition rectangles are not point like)

so ∀P this is true ∴
$$\sup_P L(f,P) \leq 0 \leq \inf_P U(f,P)$$

but as f is integrable (∵ $\sup L(f,P) = \inf U(f,P)$)
$\Rightarrow \int_A f = 0$

(b) suppose $f(x) \geq 0$ and $\int_A f = 0$

if for $x = a$ f is cont i.e
$\lim_{x \to a} f(x) = f(a)$ (defination of continuity)

if $f(a) \neq 0$ then: wlog $f(a) = \varepsilon > 0$
then by continuity f is cont at a
$\exists \delta > 0$ s.t
$f(x) > \varepsilon/2 \quad \forall x \in B(a,\delta)$
choose a partion P, of A s.t the maximum width of any subrectangle of P is less then $\delta/\sqrt{n}$

then if S_0 is a subrectangle of P which contains a, then $m_{S_0}(f) \geq \varepsilon/2$ (S_0 is non-degenrate)

Also $m_S(f) \geq 0 \quad \forall S$
$\Rightarrow L(f,P) = \sum_S m(f) v(S) \geq \varepsilon/2\, v(S_0)$

$\Rightarrow L(f,P) \leq \int_A f = 0$

$\Rightarrow \varepsilon/2\, v(S_0) \leq 0$ this is a contradiction

$\Rightarrow f(a) = 0$

so for points where f is cont $\Rightarrow f = 0$
$\Rightarrow$ f is discont on B of measure 0
$\Rightarrow$ f is 0 exept for a set of measure 0.

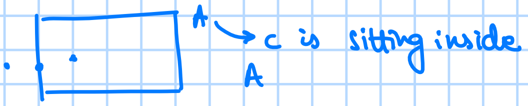
Theorm: let C be a bounded set in $\mathbb{R}^n$ and let $f: C \to \mathbb{R}$ be a bounded, cts function. let E be the set of points x_0 of Boundry (C) for which $\lim_{x \to x_0} f(x) = 0$ fail to hold.

Boundry of C

$E = \{x_0 \in B(C) \mid \lim_{x \to x_0} f(x) \neq 0\}$

If E has measure 0 then f is integrable on C.

Proof: let $x_0 \in \mathbb{R}^n \setminus E$. we show that the function $f \cdot \chi_C$ is cont at x_0



$\Rightarrow$ $x_0 \in \text{int}(C)$ then f and $f \cdot \chi_C$ agree in a nbd of x_0 as f is cts

$\Rightarrow f \cdot \chi_C$ cont on x_0

if $x_0 \in$ exterior(C) then $f \cdot \chi_C = 0$ on a nbd of x_0
$\Rightarrow$ cont on x_0

if $x_0 \in$ Boundry(C) and as $x_0 \notin E \Rightarrow f(x) \to 0$ as x approaches x_0 through points of C.

Also since $(f \cdot \chi_C)(x)$ equals $f(x)$ or 0, so
$(f \cdot \chi_C)(x) \to 0$
now as $x \to x_0$ through $\mathbb{R}^n$
$(f \cdot \chi_C)(x) \longrightarrow 0$

Now, if $x_0 \notin C$ then $(f \cdot \chi_C)(x_0) = 0$

& $x_0 \in C$ then $(f \cdot \chi_C)(x_0) = f(x_0) = 0$ by continuity of f

so in either case $f \chi_C$ is cts at x_0

$\Rightarrow f \cdot \chi_C$ is cont except on E & E has measure 0

$\Rightarrow \int_A f \cdot \chi_C = \int_C f$ and f is integrable on C

Recall: we defined a set $C \subset \mathbb{R}$ to be Jordan-measurable if C is bounded and Boundary(C) has measure 0, and we define

vol of C: $v(C) = \int_C 1$ (sets for which vol is defined are called Jordan-measurable or rectifiable)

Defn: (Rectifiable sets) Jordan-measurable sets are also called rectifiable sets.

(Rectifiable sets $\cong$ Jordan measurable)

Theorm: (a) (Positivity) If S is a rectifiable then $v(S) \geq 0$

(b) (Monotonicity) If S_1 and S_2 are rectificable and $S_1 \subseteq S_2$ then $v(S_1) \leq v(S_2)$

(c) (Additivity) If S_1, S_2 are rectifiable then so are $S_1 \cup S_2$ and $S_1 \cap S_2$ and
$v(S_1 \cup S_2) = v(S_1) + v(S_2) - v(S_1 \cap S_2)$

(d) Suppose S is rectifiable, then $v(S) = 0$ iff S has measure 0.

(e) If S is rectifiable, so is the set $A = \text{int}(S)$, $v(S) = v(A)$

(f) If S is rectifiable and if $f: S \to \mathbb{R}$ is bdd, cts function then f is integrable over S.

proof: (a), (b), (c) follows from properties of integral (proved early) part (d) follows from applying the first theorem today to non-neg function χ_S $\left(\int_A \chi_S = 0 \text{ \& } \chi_S \geq 0 \Leftrightarrow S \text{ has measure } 0 \right)$

part (f) follows from theorem (second one) proved today.
$\left(f : S \longrightarrow \mathbb{R} \text{ is bdd cts \& } E \text{ s.t. } \text{bds} \supseteq E \text{ has measure } 0 \text{ as Bd } S \text{ has measure } 0 \right)$

part (e) S is rectifiable $\Rightarrow$ S is bdd and has Boundary(S) = measure 0.
$\Rightarrow$ $A = \text{Int}(S) \subseteq S$ is bounded and boundary (A)
= boundary (int S)
= boundary(S)
$\Rightarrow$ Boundary(A) has measure 0
$\Rightarrow$ A is rectifiable.

now $v(S) = \int_S 1 = \int_Q \chi_S$ $\left(A \text{ is bounded and has measure } 0 \right)$

$Q \rightarrow$ some rectangle cont S

$v(A) = \int_Q \chi_A$ $\left(S - A = \text{Bd}(S) \right)$

$v(S) - v(A) = \int_Q (\chi_S - \chi_A)$ now

$\chi_S - \chi_A \geq 0$

and vanishes outside Boundary(S) of measure 0

$\Rightarrow \int_Q \chi_S - \chi_A = 0$

$\left(\because \int_{S-A} 1 = \int_{\text{Bd}(S)} 1 = 0 \right)$

$\Rightarrow \int_Q \chi_S = \int_Q \chi_A$

$\Rightarrow \int_S 1 = \int_A 1$ $\left(\because \int_{S-A} 1 = 0 \right)$

$\Rightarrow v(S) = v(A)$

Defn: let A be an open set in $\mathbb{R}^n$, let $f : A \longrightarrow \mathbb{R}$ be a continuous function
- If $f \geq 0$ on A we define (extended) integral of f over A, denoted by $\int_A f$ to be the supremum of $\int_D f$ as D ranges all compact, rectifiable subsets of A, provided supremum exist. In this case we say f is integrable over A. (in the extended case)
- If f is arbitrary continuous function on A then

$\left(\text{In 1-D } \int_{-\infty}^{\infty} f(x)dx = \lim_{N \to \infty} \int_{-N}^{N} f(x)dx \right)$

$f_+(x) = \max\{f(x), 0\}$
$f_-(x) = \max\{-f(x), 0\}$
as both are non-negative and cts.

f is integrable over A (in extended sense) if both f_+ and f_- are integrable on A over extended sense and in this case $\int_A f = \int_A f_+ - \int_A f_-$

11th march:

Recap: Rectifiable sets $\cong$ Jordan measurable sets (Bdd and Bd of set has measure 0)

Properties of rectifiable sets

Extended Integral, If $f \underset{cts}{\geq} 0$, A open, $\int_A f = \sup_D \int_D f$ if f is arbitrary

$D \downarrow$ compact, retifiable subset of A

$\int_A f = \int_A f_+ - \int_A f_-$

lemma: let A be an open set in $\mathbb{R}^n$, then $\exists$ a sequence $C_1, C_2, \ldots\ldots$ of compact, rectifiable subsets of A, whose union is A. s.t $C_n \subseteq int(C_{n+1})$ $\forall n \in \mathbb{N}$

proof: let $d(x,y) = |x-y|$ for $x, y \in \mathbb{R}^n$, this is a metric

for $B \subseteq \mathbb{R}^n$, define distance from x to be B

$d(x, B) = \inf\{ d(x,b) \mid \forall b \in B\}$

then $d(x,B)$ is a continious function of x

as $O(f, x_0) = \lim_{\delta \to 0} M(f, x_0, \delta) - m(f, x_0, \delta)$

$\downarrow$

let $x_1 \in |x - x_0| < \delta$

$x_2 \in |x - x_0| < \delta$

s.t $M(f, x_0, \delta) = d(x_1, B)$

$m(f, x_0, \delta) = d(x_2, B)$

now $d(x_1, B) - d(x_2, B)$

$= \inf\{d(x_1, b) \mid \forall b \in B\}$

$- \inf\{d(x_2, b) \mid \forall b \in B\}$

$\leq \inf\{ d(x_1, b) - d(x_2, b)\}$

$\leq d(x_1, x_2)$

$< \delta$

as $\delta \to 0$

$O(f, x) \to 0$

so $f(x) = d(x, B)$ is a cont fuction

now, $B = \mathbb{R}^n \setminus A$

then for $N \in \mathbb{N}$, let $D_N = \left\{ x \,\middle|\, d(x,B) \geq \frac{1}{N}, d(x,0) \leq N \right\}$

$\forall x \in D_N$, $d(x,B) \geq \frac{1}{N} \Rightarrow x \in B^c \Rightarrow x \in A$

$\Rightarrow D_N \subseteq A$ $\forall N \in \mathbb{N}$

and as $d(x,B), d(x,0)$ are continious function for x

$D_N^c = \{x \mid d(x,B) < \frac{1}{N}, d(x,0) > N\}$ is open

as $d(x,B), d(x,0)$ are cts and so D_N is closed. Also as $D_N \subseteq A \Rightarrow D_N$ is bounded

from close and Bounded $\Rightarrow D_N$ is compact

now to show $\{D_N\}$ covers A, let $x \in A$, as A is open

$\exists \delta > 0$ s.t $d(x, \delta) \subseteq A$ or $d(x,B) > 0$

$\Rightarrow \exists N \in \mathbb{N}$ s.t

$d(x,B) \geq \frac{1}{N}$ & $d(x,0) \leq N$

$\Rightarrow x \in D_N$ for some $N \in \mathbb{N}$

$\therefore \forall x \in A, \exists N \in \mathbb{N}$ s.t

so $x \in D_N$

so $A \subseteq \bigcup_{N=1} D_N$ (Aleay showed $\bigcup D_N \subseteq A$ $\Rightarrow \bigcup D_N = A$)

now let $A_{N+1} = \left\{ x \,\middle|\, d(x,B) > \frac{1}{N+1}, d(x,0) < N+1 \right\}$

this set is open

& $D_N \subset A_{N+1}$

as A_{N+1} is open

$(\text{int}(A_{N+1}) \subseteq \text{int}(D_{N+1})$

$\Rightarrow A_{N+1} \subseteq \text{int}(D_{N+1})$

so $D_N \subset A_{N+1} \subseteq \text{int}(D_{N+1})$

$\Rightarrow D_N \subset \text{int}(D_{N+1})$

now for each $x \in D_N$ choose a closed non-degerate cube that is centered at x and is contained in $\text{int}(D_N)$
the interior of cubes will cover D_N
choose finitely many of this cubes and
let their union be C_N. C_N is finite union of cubes
∴ C_N is compact and rectifiable.

$\left(\begin{array}{l}\because \cup D_N = A \\ \Rightarrow \cup C_N = A\end{array}\right)$

$D_N \subset \text{int}(C_N) \subset C_N \subset \text{int}(D_{N+1})$

$\Rightarrow \bigcup_{N=1}^{\infty} C_N = A$ & $C_N \subset \text{Int}(C_{N+1}) \; \forall N \in \mathbb{N}$

(with D_N we didnt get rectifiable)

Theorm: let A be open and in $\mathbb{R}^n$. let $f : A \longrightarrow \mathbb{R}$ be continious. choose a sequence C_N of compact rectifiable subsets of A whose union is A s.t $C_N \subset \text{int}(C_{N+1}) \; \forall N \in \mathbb{N}$. Then:

f is integrable over A $\Leftrightarrow$ subseq $\int_{C_N} |f|$ is bounded

also in this case $\int_A f = \lim_{N \to \infty} \int_{C_N} f$

also we know that f is integrable iff $|f|$ is integrable

proof: Case I: $f \geq 0$

here $f = |f|$

and as $\int_{C_N} |f|$ is increasing (monotonically)

$\left\{ \int_{C_N} |f| \right\}$ is bounded $\Leftrightarrow$ $\left\{ \int_{C_N} |f| \right\}$ is convergent

(⇒) Now as f is integrable

$$\int_A f = \sup_D \int_D f \geq \int_{C_N} f$$

$D \to$ any compact rectifiable set

$C_N \to$ our C_N

$\Rightarrow \int_{C_N} f \leq \int_A f \; \forall N \in \mathbb{N}$

$\Rightarrow \left\{ \int_{C_N} f \right\}$ is bounded and

$$\lim_{n \to \infty} \int_{C_n} f \leq \int_A f$$

(⇐) Now let $\left\{ \int_{C_N} f \right\}$ be bounded, then let D be a compact, rectifiable subset of A

as $\{C_N\}$ cover $A \Rightarrow \{C_{N_1}, C_{N_2}, \dots C_{N_r}\}$ cover D

and so $D \subseteq C_{N_1} \cup C_{N_2} \cup \cdots C_{N_r}$

also let $N = \max\{N_1, N_2, \dots N_r\}$

then

as $C_{N_i} \subseteq \text{int}(C_N) \; \forall i = 1, 2, 3, \dots r$

$\Rightarrow D \subseteq C_N$

$\Rightarrow \int_D f \leq \int_{C_N} f$ ($\because$ properties of integration)

$\Rightarrow \int_D f \leq \int_{C_N} f \leq \lim_{N \to \infty} \int_{C_N} f$

as D is arbitrary

$$\sup_{D} \int_{D} f \leqslant \lim_{n\to\infty} \int_{C_n} f$$

$\Rightarrow$ f is integrable and

$$\int_{A} f \leqslant \lim_{n\to\infty} \int_{C_n} f$$

so $\displaystyle\int_{A} f = \lim_{n\to\infty} \int_{C_n} f$

case 2 : $f : A \longrightarrow \mathbb{R}$ is an arbitary continious fuction
then f is integrable on A
iff f_+ , f_- are integrable on A
$\Updownarrow$
$\int_{C_N} f_+ , \int_{C_N} f_-$ are bounded (case I)

also $0 \leqslant f_+(x) \leqslant |f(x)|$

$0 \leqslant f_-(x) \leqslant |f(x)|$

& $|f(x)| = f_+(x) + f_-(x)$

so $\int_{C_N} f_+ , \int_{C_N} f_-$ are bounded iff $\int_{C_N} |f|$ is bounded

when $\displaystyle\lim_{n\to\infty} \int_{C_N} f_+ = \int_{A} f_+$

$$\lim_{n\to\infty} \int_{C_N} f_- = \int_{A} f_-$$

$$\lim_{n\to\infty} \int_{C_N} f = \int_{C_N} f_+ - f_- = \int_{A} f_+ - \int_{A} f_- = \int_{A} f$$

Defn : (Support of ϕ) If $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$, support is defined to be closure of $\{ x \mid \phi(x) \neq 0 \}$

(If $x \notin$ support of ϕ then $\exists$ open nbd of x s.t $\phi \equiv 0$)

Note : The proof of existance of partion of unity shows that

① we can take Φ to be coutable say $\{\phi_1, \phi_2, \dots\}$

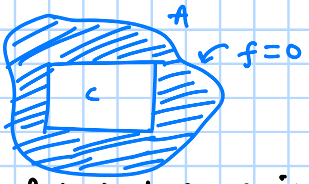
② $\forall\ \phi_i \in \Phi$, $S_i = \text{support}(\phi_i)$ is compact
(when we call this that partion of unity has compact support)

(Next we will see the conection b/w partion of unity and extended integral)

13th march :

Recap: Partition of unity, Extended region of integral

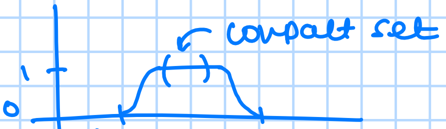
lemma : let A be open in $\mathbb{R}^n$. let $f : A \to \mathbb{R}$ be continious. If f vanishes outside a compact subset $C \subset A$, then $\int_A f$ and $\int_C f$ exist and are equal.



proof : proof of this theorm is in Munkres and will be skipped

Theorm : let A be open in $\mathbb{R}^n$. let $f : A \to \mathbb{R}$ be continious. let $\{\phi_i\}_{i \in I}$ be a partion of unity on A, having compact supports. Then the integral $\int_A f$ exist iff $\sum_{i=1}^{\infty} \int_A \phi_i |f|$ converges. In this case $\int_A f = \sum_{i=1}^{\infty} \left(\int_A \phi_i f \right)$

here support (ϕ) = closure of set $\{x \mid \phi(x) \neq 0\}$



the partion of unity we con satisfies the support (ϕ_i) is compact.

proof : proof of this will be skipped, refer munkres.

Note : $\int_A \phi_i f$ exist and equals to $\int_{S_i} \phi_i f$ when $S_i = \text{support}(\phi_i)$

Change of variables :

let's start with the 1-variable version

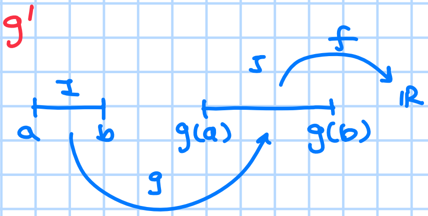
Recall that if f is integrable over [a,b], then $\int_a^b f = -\int_b^a f$

Theorm : (Substitution rule) let $I = [a,b]$, let $g : I \to \mathbb{R}$ be a function of C^1 with $g'(x) \neq 0$ for $x \in (a,b)$, then $g(I)$ is a closed interval J with endpoints $g(a)$ & $g(b)$. If $f : J \to \mathbb{R}$ is cts then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'$$

or

$$\int_J f = \int_I (f \circ g) |g'|$$



Exe : consider $\int_0^1 (2x^2+1)^{10} (4x)\, dx$

$g(x) = 2x^2+1$

$f(y) = y^{10}$

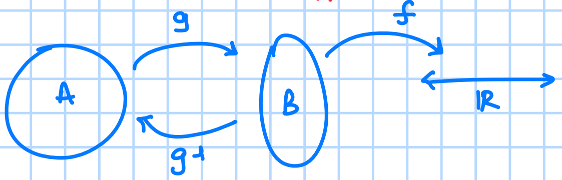
$$\int_I (f \circ g) g' = \int_{a=0}^{b=1} (2x^2+1)^{10} (4x)\, dx$$

$$= \int_{g(0)}^{g(1)} y^{10} dy = \int_1^3 y^{10} dy = \left| \frac{y^{11}}{11} \right|_1^3$$

Defn: If A and B are open sets in $\mathbb{R}^n$, and if $g: A \longrightarrow B$ is a one-to-one and onto function s.t both g & g^{-1} are C^r then g is called diffeomorphism of class C^r.

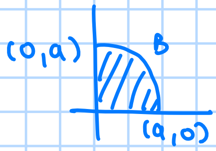
Theorm: (Change of variables) let $g: A \longrightarrow B$ be a diffeomorphism of open sets in $\mathbb{R}^n$. let $f: B \longrightarrow \mathbb{R}$ be a continuous function. Then f is integrable over B iff $(f\circ g)|\det g'|$ is integrable over A and in this case

$$\int_B f = \int_A (f\circ g)\,|\det g'|$$



Exe: ① let B be open set in $\mathbb{R}^2$ defined by

$$B = \{(x,y) \mid x>0,\ y>0,\ x^2+y^2<a\}$$

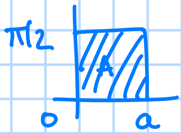


we want to compute $\int_B x^2y^2$

$g: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$g(r,\theta) = (r\cos\theta,\ r\sin\theta)$

$$g' = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}_{2\times 2}$$



$\det g' = r$

$A = \{(r,\theta) \mid 0<r<a,\ 0<\theta<\pi/2\}$

now $g: A \longrightarrow B$ is 1-1, onto and $\det g' = r > 0$

$\Rightarrow g: A \to B$ is diffeomorphism

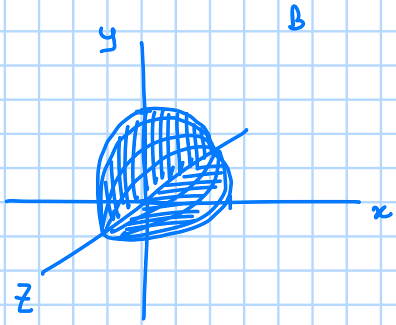
(∵ inverse function theorm)

so $\int_B f = \int_A (f\circ g)|\det g'|$

$$\int_B x^2y^2 = \int_A \underbrace{(r\cos\theta)^2(r\sin\theta)^2}_{f\circ g}\ \underbrace{r}_{|\det g'|}$$ → using fubinis we get result

② let B be the open set in $\mathbb{R}^3$, defined by $B = \{(x,y,z) \mid x>0,\ y>0,\ x^2+y^2+z^2<a^2\}$

$\int_B x^2 z$ using spherical coordinates



$g(\rho,\theta,\phi) = (\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)$

$$g'(\rho,\theta,\phi) = \begin{bmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{bmatrix}_{3\times 3}$$

$= \rho^2\sin\phi$

so $\rho^2\sin\phi > 0$ for $0<\phi<\pi$, $\rho \neq 0$

$$A = \left\{(\rho,\phi,\theta) \mid 0<\rho<a,\ 0<\phi<\pi,\ 0<\theta<\frac{\pi}{2}\right\}$$

now $g : A \longrightarrow B$ is 1-1, onto and $\det g' > 0$ on A

$$\int_B x^2 z = \int_A \underbrace{(\rho\sin\phi\cos\theta)^2(\rho\cos\phi)}_{f\circ g}\ \underset{|\det g'|}{\rho^2\sin\phi} \to \text{Fubini used then}$$

17^{th} March:

Theorm: (Substitution rule) let $I = [a,b]$, let $g : I \longrightarrow \mathbb{R}$ be a function of C' with $g'(x) \neq 0$ for $x \in (a,b)$, then $g(I)$ is a closed interval J with endpoints $g(a)$ & $g(b)$. If $f : J \longrightarrow \mathbb{R}$ is cts then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'$$

proof:

Substitution rule : $I = [0,1]$

$g : I \rightarrow \mathbb{R}$ a C' fution

$g'(x) \neq 0 \ \forall x \in (a,b)$

$J = g(I)$

let $f : J \longrightarrow \mathbb{R}$ cts

then

$$\int_J f = \int_I f \circ g |g'|$$

wlog: $g'(x) > 0$ on (a,b)

$g(b) > g(a)$, $J = [g(a), g(b)]$

$$\int_J f = \int_{g(a)}^{g(b)} f = F(g(b)) - F(g(a))$$

$F'(x) = f(x)$

$$\Rightarrow (F \circ g)'(y) = F'(g(y)) \cdot g'(y)$$
$$= f \circ g(y) \cdot g'(y)$$

now $(F \circ g)(b) - (F \circ g)(a)$

$$= \int_a^b (F \circ g)'(y) = \int_a^b (f \circ g)(g'(y))$$

Defn: Say $h : U \rightarrow V$ is a diffeomorphism of open sets in $\mathbb{R}^n$ is primitive if it keeps some coordinate fixed.

i.e

$h(x) = (h^1(x), \ldots h^n(x))$

$\exists i$ s.t

$h^i(x) = x^i$

Theorm: Any diffeomorphism $g : U \rightarrow V$ of open sets in $\mathbb{R}^n$ can be factored (in some nbd) as a composition of primitive diff.

i.e

$$\begin{tikzcd} U_0 \arrow[r, "h_1"] \arrow[d, phantom, "\cap"] & U_1 \arrow[r, "h_2"] & U_2 \arrow[r, "h_3"] & U_3 \cdots \arrow[r, "h_k"] & U_k \arrow[d, phantom, "\cap"] \\ U & & & & V \end{tikzcd}$$

$(n \geq 2)$

$$g|_{U_0} = h_k \circ h_{k-1} \circ \cdots h_1$$

proof: case I : when g is a linear transformation show how $E_{ij}(\lambda)$ is primitive or composition of primitive

now if $E_{ij}(\lambda)$ then

$$E_{ij}(\lambda) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 \cdots & \cdot 1 \cdot & \cdot \lambda & 0 \end{bmatrix} \quad i \neq j$$

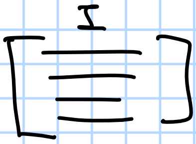
→ so $\exists$ a row s.t row is e_i so x^i as it is

→ λ at ij^{th} place

for $\begin{bmatrix} 1 & 0 & \cdot & \cdots & \\ 0 & 1 & \ddots & & 0 \\ \vdots & & & & \\ 0 & 0 & & \lambda & \\ & & & & \ddots 1 \end{bmatrix}$ then also same

λ at i,i

now $P_{ij} \rightarrow$



↕ But interchange

so same $\therefore$ $g = \Pi E_{ij}(\lambda) P_{ij}$ so g is also composition of primitive diff

Case II: g is a translation: $g(x) = x + a$ for some a

this is a trivial case

as $g(x) = x + a$
$a \in \mathbb{R}^n$
$x \in \mathbb{R}^n$

then $x + a = f \circ h$ s.t. $\underbrace{f \circ h} = g$

$$f(h(x^1, x^2, \ldots x^n))$$

$$= f(x^1 + a^1, x^2 + a^2, \ldots x^{n-1} + a^{n-1}, x^n)$$

$$= (x^1 + a^1, x^2 + a^2, \ldots x^{n-1} + a^{n-1}, x^n + a^n)$$

so $h = x + \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^{n-1} \\ 0 \end{pmatrix}$ → diff and prim (as C^r and det $\neq 0$)

$f = x + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ a^n \end{pmatrix}$ → diff and prim (as C^r and det $\neq 0$)

Case III: g is a diff s.t. $g(0) = 0$ and $Dg(0) = I$

this case $g = h \circ k$ s.t.

$h(x) = (g^1(x), \ldots g^{n-1}(x), x^n)$

$k(y) = (y^1, y^2, \ldots y^{n-1}, g^n \circ h^{-1}(y))$

Both are primitive

then $h : (\underset{C^r}{\underset{\downarrow}{g^1(x)}}, \ldots \underset{C^r}{\underset{\downarrow}{g^{n-1}(x)}}, \underset{C^r}{\underset{\downarrow}{x^n}})$

also h is one-one & onto from inverse function theorm

Same for k

now $g = h \circ k$

$$h \circ k = h(k(x))$$
$$= h(x^1, x^2, x^3, \dots x^{n-1}, g^n \circ h^{-1}(x))$$
now $$= (g^1(x), g^2(x), \dots g^n(x))$$
$$= g$$

case IV : (general case)
pre / post - compose g with traslations and liner trasformations to bring it to case 3.

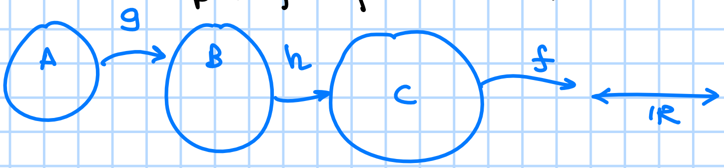
lemma : If $g : A \to B$ is a diffeomorphism of open sets in $\mathbb{R}^n$. Then for every continious fution $f : B \to \mathbb{R}$ which is integral over B

$\Rightarrow (f \circ g)|\det g'|$ is integrable over A and

$$\int_B f = \int_A (f \circ g)|\det g'|$$

proof : step I : If lemma true for diffeomorphisms $g : A \to B$ and $h : B \to C$ then it holds for $h \circ g : A \to C$
$\downarrow$
diffeomorphism

proof of this step uses chain rule



By asumption of lemma holds for h & g :

$f : C \to \mathbb{R}$ is integrable on C
then
$(f \circ h)|\det h'| : B \to \mathbb{R}$
is integrablee

& $$\int_C f = \int_B (f \circ h)|\det h'|$$

and $(f \circ h)|\det h'| : B \to \mathbb{R}$
is cts as f, h are cts

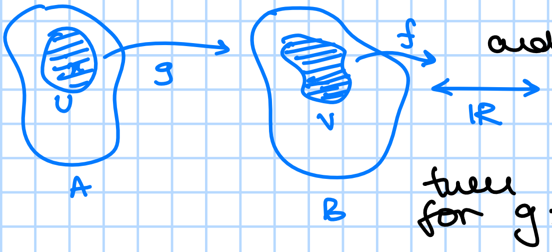
so by asumption : $((f \circ h)|\det h'|) \circ g |\det g'| : A \to \mathbb{R}$
is integrable on A

$$\int_B (f \circ h)|\det h'| = \int_A ((f \circ h)|\det h'|) \circ g |\det g'|$$

$$= \int_A (f \circ h \circ g)|\det(h'(g))||\det g'|$$

$$= \int_A f \circ (h \circ g)|\det(h \circ g)'|$$

Step II : Suppose $\forall x \in A$, $\exists$ nbd $U \ni x$ and $U \subseteq A$ s.t the lemma holds for $g : U \to V = g(U)$



and all cts $f : V \to \mathbb{R}$ s.t support of f is a compact subset of V

then we show that lemma holds for $g : A \to B$

let $\{U_\alpha\}$ is a cover of A for the above type

& let $\{\phi_i\}$ be partion of unity for B (compact Supports and subordinate to $\{V_\alpha\}$) $V_\alpha := g(U_\alpha)$

$\{\phi_i \circ g\}$ is a partiton of unity for A, with compact supports, subordinate to cover $\{U_\alpha\}$

Now, suppose $f : B \to \mathbb{R}$ is cts, integrable over B

$$\Rightarrow \int_B f = \sum_{i \geq 1} \left(\int_B \phi_i \cdot f \right)$$

$$= \sum_{i \geq 1} \int_{T_i} \phi_i \cdot f$$

$(T_i = \text{support}(\phi_i))$

$(S_i = \text{support}(\phi_i \circ g))$

now $$\int_{T_i} \phi_i \cdot f \overset{\text{lemma}}{=} \int_{V_\alpha} \phi_i f \overset{\text{By assumptions}}{=} \int_{U_\alpha} (\phi_i \cdot f) \circ g \, |\det g'|$$ (By chain rule)

$$= \int_{S_i} ((\phi_i \cdot f) \circ g \, |\det g'|$$

$$= \int_A (\phi_i \circ g)(f \circ g) \, |\det g'|$$

$$\Rightarrow \int_B f = \sum_{i \geq 1} \int_A (\phi_i \circ g)(f \circ g) \, |\det g'|$$

$$= \int_A (f \circ g) \, |\det g'|$$ exist and equals

Step 3 : Base case, lemma holds for $n=1$ details needed but follows from sub. rule

Step 4 : $n > 1$, then In order to prove lemma for arbitrary diffeomorphism, it is enough to prove for primitive diffeomorphism as from step1 & any diffeomorphism can be written as primitive ones & can be written $\circ$ U, V case II

step 5: lemma holds in dim $n-1$, then it holds in dimm n

step 4 ad 5 in next class

18th march :

lemma : let $g: A \to B$ diffo of open sets in $\mathbb{R}^n$. Then $\forall$ cts $f: B \to \mathbb{R}$ which is integ over $(f\circ g)|\det g'|$ is integrable over A, and

$$\int_B f = \int_A (f\circ g)\,|\det g'|$$

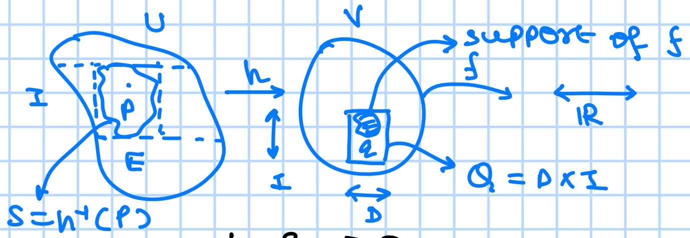
proof: step 5: Induction step, if lemma holds in $n-1$ dim then it holds in dimension n.

from 4, it is enough to prove it for primitive diffeomorphisms
let

$h: U \to V$ primitive diffeomorphism

wlog $h(\underset{\mathbb{R}^{n-1}}{x},\underset{\in\mathbb{R}}{t}) = (\underbrace{K(x,t)}_{\text{some } \mathbb{R}^{n-1}}, t)$ → primitive diff preserving last coordinate

Step 2 tells us that it is enough to prove lemma in case II conditions:



$h: S \to Q$ and
$f: Q \to \mathbb{R}$, has compact support $\subseteq Q$

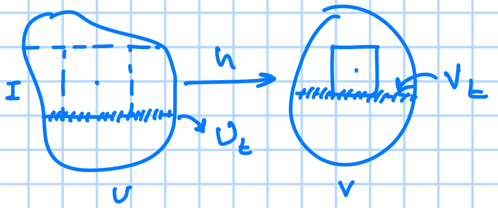
If $F = (f\circ g)|\det g'|$
then we want to show:
$$\int_Q f = \int_S F$$

Enough to show $\int_Q f = \int_{E\times I} F$

$$\int_{t\in I}\int_{y\in D} f(y,t) = \int_{t\in I}\int_{x\in E} F(x,t)$$

↳ Replace by V_t Replace by U_t

I are same as primitive diffeomorphism



now, $Dh = \begin{bmatrix} \partial K/\partial x & \partial K/\partial t \\ 0 & 1 \end{bmatrix}$

$\det h' \neq 0 \Rightarrow \det\left(\frac{\partial K}{\partial x}\right) \neq 0$

so $x \longmapsto K(x,t)$ for fixed t is a diff from U_t to V_t
$(\mathbb{R}^{n-1})$

so we can apply induction hypothesis and get

$$\int_{y\in V_t} f(y,t) = \int_{x\in U_t} F(x,t)$$

$$= \int_{x\in U_t} f(k(x,t),t)\,|\det \frac{\partial k}{\partial x}|$$

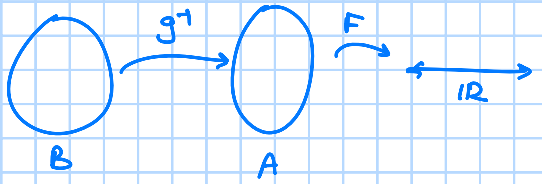
$$= \int_{x\in U_t} (f\circ h)\,|\det h'|$$

Lemma: let $g: A \to B$ be a diffeomorphism of open sets in $\mathbb{R}^n$. $f: B \to \mathbb{R}$ cts
If $(f\circ g)|\det g'|$ is integrable over A, then f is integrable over B.

proof:

let $F = (f\circ g)|\det g'|$
F is cts, integrable over A then

(The proof is just inverse of the previous one)



g^{-1} is also a diffeomorphism

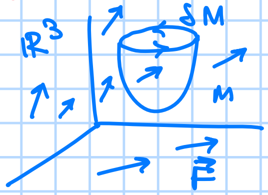
so the previous lemma tells us that

$(F\circ g^{-1})|\det(g^{-1})'|$ is integrable over B

$= (f\circ g\circ g^{-1})|\cancel{\det g'}||\cancel{\det(g^{-1})'}|$

$= f$ → so f is integrable over B

goal: n-dimmension version of stokes theorm



$$\iint_M (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dA = \oint_{\delta M} \vec{F}\cdot d\vec{s}$$

cross product in $\mathbb{R}^3$:

$(\vec{v}_1 + \vec{v}_2)\times\vec{w} = \vec{v}_1\times\vec{w} + \vec{v}_2\times\vec{w}$ → property of cross product

there are other properties aswell

Multilinear algebra:

let V be a vectorspace of dimm n on $\mathbb{R}$. write $V^k = \underbrace{V\times V\times\cdots V}_{k \text{ times}}$

Defn: A function $T: V^k \to \mathbb{R}$ is said to be multilinear if it is seperately linear with all of its coordinates.

i.e $\forall i,\ 1\le i\le k$

$T(v_1,\dots v_i+w_i,\dots v_k) = T(v_1,v_2\dots v_i\dots v_k) + T(v_1,v_2,w_i,\dots)$

$T(v_1,\dots a v_i,\dots v_k) = a\,T(v_1,v_2,\dots v_i,\dots)$

Eg: $\langle\bullet,\bullet\rangle: V\times V \to \mathbb{R}$

$(v,w) \to \langle v,w\rangle$

this is multilinear

Eg : det : $M_{n\times n}(\mathbb{R}) \longrightarrow \mathbb{R}$

can be thought of as det : $V^n \longrightarrow \mathbb{R}$

$V=\mathbb{R}^n$

$\det(v_1,\ldots v_n) = \det[v_1\ v_2 \ldots v_n]$

then det : $V^n \longrightarrow \mathbb{R}$ is multilinear

Defn : (Tensor) A multilinear funtion $T : V^k \rightarrow \mathbb{R}$ is called a k-tensor on V

The set of all k tensors on V is denoted by $T^k(V)$

Eg: set of all 1-Tensors $T^1(V)$ is s.t

$\forall\ T \in T^1(V)$

$T : V \longrightarrow \mathbb{R}$ is multilinear

$\Rightarrow$ linear

so $T \in V^*$ (dual space)

so

$T^1(V) = V^*$

For $S,T \in T^k(V)$ and $a \in \mathbb{R}$ define

$$(S+T)(v_1,\ldots v_k) = S(v_1,\ldots v_k) + T(v_1,\ldots v_k)$$
$$(aT)(v_1,\ldots v_k) = a\,T(v_1,\ldots v_k)$$

so $S+T,\ aT \in T^k(V)$

$\therefore$ $T^k(V)$ is a vectorspace over $\mathbb{R}$

Defn : (Tensor product) If $S \in T^k(V)$ and $T \in T^l(V)$ then define the tensor product $S\otimes T \in T^{k+l}(V)$ by

$$(S\otimes T)(v_1,\ldots v_k, v_{k+1}, \ldots v_{k+l})$$
$$= S(v_1, v_2 \ldots v_k)\cdot T(v_{k+1},\ldots v_{k+l})$$

$\otimes$ is called the tensor product

Note : In general $S\otimes T \neq T\otimes S$

properties of $\otimes$:

$$(S_1+S_2)\otimes T = S_1\otimes T + S_2 \otimes T$$
$$S\otimes(T_1+T_2) = S\otimes T_1 + S\otimes T_2$$
$$(aS)\otimes T = S\otimes(aT) = a(S\otimes T)$$
$$(S\otimes T)\otimes U = S\otimes(T\otimes U) = S\otimes T\otimes U$$

Theorm : let $\{v_1,\ldots v_n\}$ be a basis for V, let $\{\phi_1 \ldots\ldots \phi_n\}$ be the dual basis $\varphi_i(v_j) = \delta_{ij}$ then the set

$$\{\varphi_{i_1}\otimes\varphi_{i_2}\cdots\otimes\varphi_{i_k} \mid 1\leq i_1,\ldots,i_k \leq n\}$$

is basis for $T^k(V)$

In particular $\dim(T^k(V)) = \underset{\downarrow}{n}^k$

$\underbrace{n \text{ choices} \times n \text{ choices} \cdots}$ k times

proof : $(\varphi_{i_1}\otimes\varphi_{i_2}\cdots\otimes\varphi_{i_k})(v_{j_1}, v_{j_2}, \ldots v_{j_k})$

$$= (\delta_{i_1,j_1} \ldots \delta_{i_k,j_k})$$
$$= \begin{cases} 1 ; & j_r = i_r\ \forall r \\ 0 ; & \text{otheise} \end{cases}$$

let $(w_1, w_2 \dots w_k) \in V^k$ $\quad w_i = \sum_{j=1}^{n} a_{ij} v_j$ $\quad (\text{as } w_i \in V)$

let $T \in T^k(V)$

$$T(w_1, w_2 \dots w_k) = T\left(\sum a_{1j} v_j, \sum a_{2j} v_j \cdots\right)$$

$$= \sum_{j_1,\dots j_k=1}^{n} a_{1j_1} \cdots a_{kj_k} T(v_{j_1}, v_{j_2}, \dots v_{j_k})$$

where $a_{ij_i} = \varphi_{j_i}(w_i)$ $\quad \left(\begin{array}{l} w_i = \sum_{v=1}^{n} a_{ij} v_j \\ \varphi_j(w_i) = a_{ij} \varphi_j(v_j) \\ \quad = a_{ij} \end{array} \right)$

$$= \sum_{j_1,\dots j_k=1}^{n} \varphi_{j_1}(w_1) \cdots \varphi_{j_k}(w_k)\, T(v_{j_1}, \dots v_{j_k})$$

$$= \sum_{j_1 \dots j_k=1}^{n} (\varphi_{j_1} \otimes \varphi_{j_2} \otimes \cdots \otimes \varphi_{j_k})(w_1, \dots w_k) \cdot T(v_{j_1} \dots v_{j_k})$$

thus $\quad T = \sum_{j_1 \dots j_k=1}^{n} T(v_{j_1} \dots v_{j_k})(\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k})$

so $\quad \{\varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k} \mid 1 \le j_1, \dots j_k \le n\}$ spans $T^k(V)$

also this is lin ind as:

if $\exists \{a_{i_1 \dots i_k} \mid 1 \le i_1, \dots i_k \le n\}$

s.t $\sum_{i_1 \dots i_k=1}^{n} a_{i_1 \dots i_k} (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}) = 0$

apply both sides to $(v_{j_1} \dots v_{j_k}) \in V^k$

$$\sum_{i_1 \dots i_k=1}^{n} a_{i_1 \dots i_k} (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})(v_{j_1}, \dots v_{j_k}) = 0$$

$$\Rightarrow \sum_{i_1 \dots i_k=1}^{n} a_{i_1 \dots i_k} (\delta_{i_1 j_1})(\delta_{i_2 j_2}) \cdots (\delta_{i_k j_k}) = 0$$

now for $i_1 = j_1,\ i_2 = j_2 \dots\ i_k = j_k$

$$\sum_{i_1 \cdots i_k=1}^{n} a_{i_1 \dots i_k} (\delta_{i_1 j_1}) \cdots (\delta_{i_k i_k}) = a_{j_1, j_2, \dots j_k} = 0$$

$$\Rightarrow a_{j_1 \dots j_k} = 0$$

this is true $\forall$ k-tuple $(j_1 \dots j_k)$

$\Rightarrow \{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \mid 1 \le i_1 \cdots i_k \le n\}$

is linearly independent

20th march :

Recap : We finished the proof of change of variables theorem

V, a vector space over $\mathbb{R}$, defined k-tensor on V; the set of all such is denoted by $T^k(V)$

Eg : $\langle \cdot , \cdot \rangle \in T^2(V)$
$\det M_{n\times n}(\mathbb{R}) \in T^n(\mathbb{R}^n)$

Note : $T^k(V)$ is a vector space over $\mathbb{R}$

Now if $\{v_1, \dots v_n\}$ is a basis for V and $\{\varphi_1, \dots \varphi_n\}$ is the dual basis, then

$$\{ \varphi_{i_1} \otimes \varphi_{i_2} \otimes \cdots \varphi_{i_k} \mid 1 \leq i_1, \dots i_k \leq n \}$$

is a basis for $T^k(V)$

$$T^k(V) = \text{span}\{ \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} \}$$

& $\{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}\}$ is linearly independent

Exe : show that $T(x,y,z) := x^1y^2z^4 + 2x^1y^4z^3$ for $x,y,z \in \mathbb{R}^4$ is a 3-tensor on $\mathbb{R}$

Now, $T := \varphi_1 \otimes \varphi_2 \otimes \varphi_4 + 2\varphi_1 \otimes \varphi_4 \otimes \varphi_3$

$$\in \text{span}\{ \varphi_i \otimes \varphi_j \otimes \varphi_k \mid 1 \leq i,j,k \leq 4 \}$$

when $\varphi_i(e_j) = \delta_{ij}$ $\Rightarrow$ as T can be written as linear combination of Basis of $T^3(\mathbb{R}^4)$, it is a tenso

Defn : If $f: V \to W$ is a linear transformation we have another linear transformation $f^*: T^k(W) \to T^k(V)$ defined as follows:

$$T \longmapsto (f^* T)$$

$$(f^*T)(v_1, \dots v_k) := T(f(v_1), \dots f(v_k))$$

$$\underset{T^k(V)}{\overset{\cap}{(f^*T)}}(v_1, \dots v_k) = T(f(v_1) \dots) \qquad \underset{T^k(V)}{\cap} \quad \text{for } T \in T^k(W)$$

$v_1, v_2 \dots v_k \in V$, so $f^*(T) \in T^k(V)$
$f(v_i) \in W$

Note : $f^*(S \otimes T) = f^*(S) \otimes f^*(T)$
$(f^*(S \otimes T) := f^*(S \otimes T)(v_1, \dots v_k) = S \otimes T(f(v_1) \cdots f(v_k)) = f^*(S) \otimes f^*(T))$

Defn : let T be a k-tensor on V, we say T is symmetric if $\forall$ pair $i \neq j$

$$T(v_1, \dots v_j, \dots v_i, \dots v_k) = T(v_1, \dots v_i, \dots v_j, \dots v_k)$$

Eg : $\langle \cdot , \cdot \rangle$ is a symmetric 2-tensor

Defn : We say that T is alternating tensor if for any pair i, j :

$$T(v_1, \dots v_j, \dots v_i, \dots v_k) = -T(v_1, \dots v_i, \dots v_j, \dots v_k)$$

Eg : $\det$ is an alternating n-tensor on $\mathbb{R}^n$

Defn : The set of all alternating k-tensors on V is denoted by $\Lambda^k(V)$

Note : $\Lambda^k(V) \subseteq T^k(V)$ is a vector space of $T^k(V)$ ($T \in \Lambda^k(V)$ then T altering)

Defn : For $T \in T^k(V)$, we define a new k-tensor called $\text{Alt}(T)$, as follows:

$$\text{Alt}(T)(v_1, \dots v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma)\, T(v_{\sigma(1)}, \dots v_{\sigma(k)})$$

where S_k is the set of permutations of $\{1, 2 \dots n\}$

Theorm : ① If $T \in T^k(V)$ then $Alt(T) \in \Lambda^k(V)$

② if $\omega \in \Lambda^k(V)$ then $Alt(\omega) = \omega$

③ if $T \in T^k(V)$ then $Alt(Alt(T)) = Alt(T)$

proof :

① If $\sigma \in S_k$, let $\sigma' = \sigma \cdot (ij)$ then

$$(Alt\, T)(v_1, \dots v_j, \dots v_i, \dots v_k)$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \cdot T(v_{\sigma(1)}, \dots v_{\sigma(j)} \dots v_{\sigma(i)} \dots v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) T(v_{\sigma'(1)}, \dots v_{\sigma'(i)}, \dots v_{\sigma'(j)}, \dots v_{\sigma'(k)})$$

$$= \frac{1}{k!} \sum_{\sigma' \in S_k} -sgn(\sigma') T(v_{\sigma'(1)}, \dots v_{\sigma'(k)})$$

$$= -Alt(T)(v_1, \dots v_i, \dots v_j, \dots v_k)$$

$\Rightarrow Alt(T) \in \Lambda^k(V)$

② Now if $\omega \in \Lambda^k(V)$ then $\tilde{\sigma} = (ij)$

then

$$\omega(v_{\tilde{\sigma}(1)}, \dots v_{\tilde{\sigma}(k)}) = -\omega(v_1, \dots v_k)$$

$$= sgn(\tilde{\sigma}) \omega(v_1, \dots v_k)$$

now, every $\sigma \in S_k$ is a product of traspositions (ij) for some i, j

$$\Rightarrow \omega(v_{\sigma(1)} \dots v_{\sigma(k)}) = sgn(\sigma) \omega(v_1, \dots v_k)$$

this is $\forall \sigma \in S_k$, so

$$Alt(\omega)(v_1, \dots v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) \omega(v_{\sigma(1)}, \dots v_{\sigma(k)})$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} sgn(\sigma) sgn(\sigma) \omega(v_1, \dots v_k)$$

$$= \frac{1}{k!} \sum_{\sigma \in S_k} \omega(v_1, \dots v_k)$$

$$= \frac{k!}{k!} \omega(v_1, \dots v_k)$$

$\Rightarrow Alt(\omega) = \omega$

③ This follows from ①, ② as $Adj(T) \in \Lambda^k(V)$

$$\Rightarrow Adj(Adj(T)) = Adj(T)$$

Defn : For $\omega \in \Lambda^k(V)$ and $\eta \in \Lambda^\ell(V)$, the wedge product

$$\omega \wedge \eta \in \Lambda^{k+\ell}(V)$$

is defined by

$$\omega \wedge \eta = \frac{(k+\ell)!}{k!\, \ell!} Alt(\omega \otimes \eta)$$

Properties of wedge product:

$$\left(\omega \wedge \eta = \frac{(k+l)!}{(k)!(l)!} \mathrm{Alt}(\omega \otimes \eta) \right)$$

$(\omega_1 + \omega_2) \wedge \eta = (\omega_1 \wedge \eta) + (\omega_2 \wedge \eta)$

$\omega \wedge (\eta_1 + \eta_2) = (\omega \wedge \eta_1) + (\omega \wedge \eta_2)$

$a\omega \wedge \eta = \omega \wedge a\eta = a(\omega \wedge \eta)$

$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$

$f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

Theorm : ① If $S \in T^k(V)$ and $T \in T^l(V)$ and $\mathrm{Alt}(S) = 0$, then $\mathrm{Alt}(S \otimes T) = \mathrm{Alt}(T \otimes S) = 0$

② $\mathrm{Alt}(\mathrm{Alt}(\omega \otimes \eta) \otimes \theta) = \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$
$= \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta))$

proof: ① $(k+l)!\, \mathrm{Alt}(S \otimes T)(v_1, v_2 \dots v_{k+l})$

$$= \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma) S(v_{\sigma(1)}, \dots v_{\sigma(k)}) T(v_{\sigma(k+1)} \dots v_{\sigma(k+l)})$$

let $G \subset S_{k+l}$ be the subgroup consisting of all $\sigma \in S_{k+l}$ which leaves $k+1, \dots k+l$ fixed then

$$\sum_{\sigma \in G} \mathrm{sgn}(\sigma) S(v_{\sigma(1)}, \dots v_{\sigma(k)}) T(v_{\sigma(k+1)} \dots v_{\sigma(k+l)})$$
$$= \sum_{\sigma \in G} \mathrm{sgn}(\sigma) S(v_{\sigma(1)}, \dots v_{\sigma(k)}) T(v_{k+1}, \dots v_{k+l})$$
$$= \left[k!\, \mathrm{Alt}(S)(v_1 \dots v_k) \right] T(v_{k+1} \dots v_{k+l})$$
$= 0$ as $\mathrm{Alt}(S) = 0$

now, for any right coset $\sigma_0 \in S_{k+l}$

$$G \circ \sigma_0 = \{ \sigma \cdot \sigma_0 \mid \forall \sigma \in G \}$$

let $(v_{\sigma_0(1)}, \dots v_{\sigma_0(k+l)}) = (w_1, w_2 \dots w_{k+l})$

then $\displaystyle\sum_{\sigma \in G \cdot \sigma_0} \mathrm{sgn}(\sigma) S(v_{\sigma(1)} \dots v_{\sigma(k)}) T(v_{\sigma(k+1)} \dots v_{\sigma(k+l)})$

$$= \left[\mathrm{sgn}(\sigma_0) \sum_{\sigma' \in G} \mathrm{sgn}(\sigma') S(v_{\sigma'(1)} \dots v_{\sigma'(k)}) \right] T(w_{k+1}, \dots w_{k+l})$$

$$= \left[\mathrm{sgn}(\sigma_0) \cdot k!\, \mathrm{Alt}(S)(w_1 \dots w_k) \right] T(w_{k+1} \dots w_{k+l})$$

$= 0$

now, write S_{k+l} as disjoint union of right cosets breaks $\displaystyle\sum_{\sigma \in S_{k+l}} \cdots$ into sums over right cosets

that $\displaystyle\sum_{\sigma \in S_{k+l}} \cdots = 0 \Rightarrow \mathrm{Alt}(S \otimes T) = 0$

similary $\mathrm{Alt}(T \otimes S) = 0$ is similar

② $\mathrm{Alt}(\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta) = \mathrm{Alt}(\eta \otimes \theta) - \mathrm{Alt}(\eta \otimes \theta)$
$= 0$

then by part ① let $T = \omega$
$S = \mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta$

then $\mathrm{Alt}(S \otimes T) = \mathrm{Alt}(T \otimes S) = 0$

$\Rightarrow \mathrm{Alt}(\omega \otimes [\mathrm{Alt}(\eta \otimes \theta) - \eta \otimes \theta]) = 0$

$\Rightarrow \mathrm{Alt}(\omega \otimes \mathrm{Alt}(\eta \otimes \theta)) = \mathrm{Alt}(\omega \otimes \eta \otimes \theta)$

24th March:

Recap from last week: An alternating tensor $T(v_1\dots v_k) = -T(v_1\dots v_j,\dots v_i,\dots v_k)$
And set of all alternating k-tensors is a subspace of $T^k(V)$ denoted by $\Lambda^k(V)$

If $T\in T^k(V)$ then $Alt(T) = \frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma)\, T(v_{\sigma(1)},\dots v_{\sigma(k)}) \in \Lambda^k(V)$

also, wedge product $\omega\in\Lambda^k(V)$, $\eta\in\Lambda^l(V)$ then $\omega\wedge\eta = \frac{(k+l)!}{n!\,l!} Alt(\omega\otimes\eta)$

Theorem: If $S\in T^k(V)$, $T\in T^l(V)$

① if $Alt(S)=0$ then $Alt(T\otimes S) = Alt(S\otimes T) = 0$

② $Alt(Alt(\omega\otimes\eta)\otimes\theta) = Alt(\omega\otimes\eta\otimes\theta) = Alt(\omega\otimes Alt(\eta\otimes\theta))$

③ If $\omega\in\Lambda^k(V)$, $\eta\in\Lambda^l(V)$ and $\theta\in\Lambda^m(V)$ then

$$(\omega\wedge\eta)\wedge\theta = \omega\wedge(\eta\wedge\theta) = \frac{(k+l+m)!}{k!\,l!\,m!} Alt(\omega\otimes\eta\otimes\theta)$$

proof: ①,② already done, so now for ③.

$$(\omega\wedge\eta)\wedge\theta = \frac{(k+l+m)!}{(k+l)!\,m!} Alt((\omega\wedge\eta)\otimes\theta)$$

$$= \frac{(k+l+m)!}{\cancel{(k+l)}!\,m!} Alt\left(\frac{\cancel{(k+l)}!}{k!\,l!} Alt(\omega\otimes\eta)\otimes\theta\right)$$

$$= \frac{(k+l+m)!}{k!\,l!\,m!} Alt(Alt(\omega\otimes\eta)\otimes\theta)$$

By ②:

$$= \frac{(k+l+m)!}{k!\,l!\,m!} Alt(\omega\otimes\eta\otimes\theta)$$

The other equality can be proved similarly

(Now with this we can find a basis for $\Lambda^k(V)$)

Theorem: The set $\mathcal{B} = \{\varphi_{i_1}\wedge\dots\wedge\varphi_{i_k} \mid 1\le i_1<i_2\dots<i_k\le n\}$ is a basis for $\Lambda^k(V)$
In particular,

$$dim(\Lambda^k(V)) = \binom{n}{k} = \frac{n!}{(k)!\,(n-k)!}$$

Here $\Lambda^1(V) = V^*$ where $\varphi_i\in V^*$
or $\varphi_i\in\Lambda^1(V)$
so $\varphi_i\wedge\varphi_i = (-1)^{1\times 1}\varphi_i\wedge\varphi_i$
$= -\varphi_i\wedge\varphi_i$
$\Rightarrow \varphi_i\wedge\varphi_i = 0$
so we have strict inequality

proof: If $\omega\in\Lambda^k(V)\subseteq T^k(V)$, then we have

$$\omega = \sum_{i_1,\dots i_k} a_{i_1,i_2\dots i_k}\, \varphi_{i_1}\otimes\varphi_{i_2}\otimes\dots\otimes\varphi_{i_k}$$

($\because T^k(V)\ni\omega$, and as $\{\varphi_{i_1}\otimes\varphi_{i_2}\dots\otimes\varphi_{i_k} \mid 1\le i_1,\dots i_k\le n\}$ is a basis for $T^k(V)$)

for some numbers $\{a_{i_1},\dots a_{i_k}\}$, since $\omega\in\Lambda^k(V)$, we have
$\omega = Alt(\omega) = \sum_{i_1,\dots i_k} a_{i_1,\dots i_k} Alt(\varphi_{i_1}\otimes\cdots\varphi_{i_k})$

by previous theorem
$$Alt(\varphi_{i_1}\otimes\cdots\varphi_{i_k}) = \underset{\substack{\downarrow\\ \text{constant}}}{c}\cdot(\varphi_{i_1}\wedge\varphi_{i_2}\wedge\varphi_{i_3}\cdots\wedge\varphi_{i_k})$$

also $\varphi_i\wedge\varphi_j = (-1)^{1\cdot 1}\varphi_j\wedge\varphi_i = -\varphi_j\wedge\varphi_i$

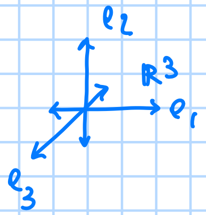
(in particular $\varphi_i \wedge \varphi_i = 0$)

so $\varphi_{i_1} \wedge \varphi_{i_2} \cdots \wedge \varphi_{i_k} \neq 0$ when all i_j's are distinct.
we can also rearrange wlog $i_1 < i_2 < \dots < i_k$
∴ w can be expressed as a linear combination of $\mathcal{B}$.

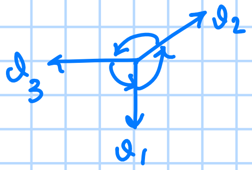
∴ $\Lambda^k(V) = \mathrm{span}(\mathcal{B})$

now to show $\mathcal{B}$ is linearly independent
if
$$\sum_{i_1,\dots i_k} a_{i_1,i_2\dots i_k}\, \varphi_{i_1} \wedge \varphi_{i_2} \dots \wedge \varphi_{i_k} = 0$$
apply $(v_{j_1}, v_{j_2} \dots v_{j_k})$ on both sides to get $a_{j_1 \dots j_k} = 0$, this is true for all $j_1, j_2 .. j_k$

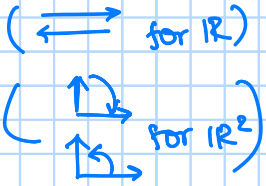
orientation of vectorspace :



$\{e_1, e_2, e_3\}$ is a right-handed basis for $\mathbb{R}^3$



$\{v_1, v_2, v_3\}$ is one more example of right handed basis.



$\{e_2, e_1, e_3\}$ is a left-handed basis

Note : $[e_1\ e_2\ e_3]$ matrix, $\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 > 0$
$\qquad\qquad\qquad e_1\ \ e_2\ \ e_3$

$$\det\,[e_2\ e_1\ e_3] := \det \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = -1 < 0$$

Now notice that if $\{v_1, v_2, v_3\}$ and $\{w_1, w_2, w_3\}$ are both right-handed basis, then if
$$A = (a_{ij}) \text{ where } w_i = \sum_{j=1}^{3} a_{ij} v_j$$
then $\det(A) > 0$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad A\,[v_1\ v_2\ v_3] = [w_1\ w_2\ w_3]$$

and so, this defines an equivalence relation on the set of all basis of $\mathbb{R}^3$, i.e $B_1 \sim B_2$ if the matrix A that transforms $B_1 \to B_2$ has positive determinant.

so there are two equivalence classes, set of all right handed basis and set of all left handed basis.

A choice of one of these two equivalence classes is called an orientation for $\mathbb{R}^3$.

25th march.

$\swarrow$ Alt space of $T^k(V)$

Recap from yesterday: let $\omega \in \Lambda^k(V)$, $\eta \in \Lambda^l(V)$, $\theta \in \Lambda^m(V)$

$\swarrow$ wedge product

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta)$$

$$\frac{(k+l+m)!}{(k)!(l)!(m)!} \, \text{Alt}(\omega \otimes \eta \otimes \theta) = \omega \wedge \eta \wedge \theta$$

- also $\mathcal{B} = \{e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1 < i_2 < \dots < i_k \leq n\}$ is a basis for $\Lambda^k(V)$

$\swarrow$ dim of V

$$\dim(\Lambda^k(V)) = {}^nC_k$$

$\downarrow$ value of k

- we also covered right handed/left handed basis for $\mathbb{R}^3$ (also for $\mathbb{R}^2$, $\mathbb{R}$)
- Also orientation for $\mathbb{R}^3 \setminus \mathbb{R}^2 \setminus \mathbb{R}$

- Equivalence relation on the set of ordered basis for $\mathbb{R}^3$.
  $\hookrightarrow$ $\{e_1, e_2, e_3\}$ is different from $\{e_2, e_1, e_3\}$

orientation of a vector space:

If $\dim(V) = n$, then $\dim(\Lambda^n(V)) = \binom{n}{n} = 1$

so all alternating n-tensor on V are multiple of any chosen (non-zero) one.

Theorm: let $v_1, \dots v_n$ be a basis for V. Let $\omega \in \Lambda^n(V)$. If $w_i = \sum_{j=1}^{n} a_{ij} v_j$ are n vectors in V then:

$$\omega(w_1, \dots w_n) = \det((a_{ij}))\, \omega(v_1, \dots v_n)$$

proof: define η (a $T^n(\mathbb{R}^n)$)
$\eta \in T^n(\mathbb{R}^n)$ is s.t

$$\eta\left(\begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \\ \vdots \\ a_{1n} \end{pmatrix}, \begin{pmatrix} a_{21} \\ a_{22} \\ \vdots \\ a_{2n} \end{pmatrix}, \dots \begin{pmatrix} a_{n1} \\ \vdots \\ a_{nn} \end{pmatrix}\right) = \omega\left(\sum_{j=1}^{n} a_{1j} v_j, \sum_{j=1}^{n} a_{2j} v_j, \dots \sum_{j=1}^{n} a_{nj} v_j\right)$$

$\left(\begin{matrix} A[v_1 \dots v_n] \\ = [w_1, \dots w_n] \end{matrix}\right)$

then $\eta \in \Lambda^n(\mathbb{R}^n)$ so

$\eta = \lambda \det$ for some λ $\left(\begin{matrix} \because \det \in \Lambda^n(\mathbb{R}^n) \\ \& \dim(\Lambda^n(\mathbb{R}^n)) = 1 \end{matrix}\right)$

$$\eta(e_1 \dots e_n) = \lambda \underbrace{\det(e_1 \dots e_n)}_{= \lambda}$$

$$\Rightarrow \lambda = \eta(e_1 \dots e_n)$$

$$\Rightarrow \lambda = \omega(v_1, v_2 \dots v_n)$$

(By defn $\eta(e_1, e_2 \dots e_n) = \omega(v_1, \dots v_n)$)
$\lambda = \omega(v_1, \dots v_n)$
$\lambda \det((a_{ij})) = \omega(w_1, \dots w_n)$

so $\omega(\sum a_{1j} v_j, \dots \sum a_{nj} v_j)$

$$= \omega(w_1, \dots w_n)$$

so $\omega(w_1, \dots w_n) = \det((a_{ij}))\, \omega(v_1 \dots v_n)$

Note: Now if $\omega \in \Lambda^n(V)$ with $\omega \neq 0$ and $\{v_1, \dots v_n\}$ is a basis for V then $\omega(v_1, \dots v_n) \neq 0$

so every basis for V belongs to one of two groups:

Group I: set of all basis $\{v_1, \dots v_n\}$ s.t

$$\omega(v_1, \dots v_n) > 0$$

Group II: The set of all basis $\{v_1, \dots v_n\}$ s.t $\omega(v_1, \dots v_n) < 0$

Now let $\{v_1, \dots v_n\}$, $\{w_1, \dots w_n\}$ be two basis for V and let

$$A = (a_{ij}) \text{ matrix}$$
$$\text{by} \quad w_i = \sum_{j=1}^{n} a_{ij} v_j$$

then by theorm proved,
$\{v_1, \dots v_n\}$ & $\{w_1, \dots w_n\}$ are
in the same group if
$\det((a_{ij})) > 0$

$\therefore$ we have obtained a criterion to seprate the basis V into 2 groups independent of the choice of w. (we can drop w now)

Each of this group is called an <u>orientation</u> for vector space V.

<u>Defn</u>: The orientation to which given basis $\{v_1, \dots v_n\}$ belongs is denoted by $[v_1, \dots v_n]$ and the other orientation is denoted by $-[v_1, \dots v_n]$

<u>Volume element of V</u>:

<u>Lemma</u>: let V be a vector space with inner product T. let $\{v_1, \dots v_n\}$ and $\{w_1, \dots w_n\}$ be two basis for V which are orthonormal wrt T. let

$$A = (a_{ij})$$
$$w_i = \sum_{j=1}^{n} a_{ij} v_j$$

then $AA^T = I$

<u>proof</u>:

$$\delta_{ij} = T(w_i, w_j) = T(\sum a_{ik} v_k, \sum a_{jl} v_l)$$
$$= \sum_{k,l=1}^{n} a_{ik} a_{jl} T(v_k, v_l)$$
$$= \sum_{l=1}^{n} a_{il} a_{jl} = (AA^T)_{ij}$$
$$\delta_{ij} = (AA^T)_{ij}$$
$$\Rightarrow AA^T = I$$

<u>Note</u>: In the above case, $\det(A) = \pm 1$
Therefore if $\omega \in \Lambda^n(V)$ s.t
$\omega(v_1, \dots v_n) = \pm 1$
then $\omega(w_1, \dots w_n) = \pm \det(A) = \pm 1$

<u>Defn</u>: let μ be orientation for V, then $\exists$ unique $\omega \in \Lambda^n(V)$ s.t $\omega(v_1, \dots v_n) = 1$ whenever $\{v_1, \dots v_n\}$ is an ortonormal basis with $\mu = [v_1, \dots v_n]$. This unique ω is called the volume element of V, determined by the Inner product T and the orientation μ.

<u>Cross product</u>: ($\exists$ unique ω wrt orthonormal basis, μ, T)
$\omega(v_1, \dots v_n) = 1$ ↳ vol ele

let $v_1, \dots v_{n-1} \in \mathbb{R}^n$, define $\varphi \in \Lambda^1(\mathbb{R}^n)$ $(\Lambda^1(\mathbb{R}^n) \leftrightarrow (\mathbb{R}^n)^*)$
as follows:

For $w \in \mathbb{R}^n$, $\varphi(w) := \det \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \\ w \end{bmatrix} \overset{\curvearrowleft}{} \text{row}$ now using the inner product $\langle \cdot, \cdot \rangle$

we have a vector space isomorphism $\left(\varphi \in (\mathbb{R}^n)^*\right)$

$V = \mathbb{R}^n$ (in this case)

$F : V \longrightarrow V^*$ given by $\left(\det \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{n-1} \\ \omega \end{bmatrix} = \varphi(\omega)\right)$

$v \longmapsto \varphi_v$

where $\varphi_v(\omega) := \langle v, \omega \rangle$

Let $z \in \mathbb{R}^n$, there unique vector satisfying $\varphi = \varphi_z$

i.e

$$\langle z, \omega \rangle = \varphi(\omega) = \det \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \\ \omega \end{bmatrix}$$

Defn: $z = v_1 \times v_2 \ldots \times v_{n-1}$ and call z the cross product of the vector $v_1, v_2 \ldots v_{n-1}$

when $= 3$ we have $v_1, v_2 \in \mathbb{R}^3$ (it matches with usual cross

$$a\hat{i} + b\hat{j} + c\hat{k} = v_1$$
$$d\hat{i} + e\hat{j} + f\hat{k} = v_2$$
$$\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k} = \omega$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ \omega_1 & \omega_2 & \omega_3 \end{bmatrix} = z_1\omega_1 + z_2\omega_2 + z_3\omega_3$$

$$\Rightarrow ae\omega_3 + d\omega_2 c + bf\omega_1 - ce\omega_1 - fa\omega_2 - bd\omega_3 = z_1\omega_1 + z_2\omega_2 + z_3\omega_3$$

$$\Rightarrow \begin{aligned} bf - ce &= z_1 \\ dc - fa &= z_2 \\ ae - bd &= z_3 \end{aligned}$$

so $(a, b, c) \times (d, e, f)$

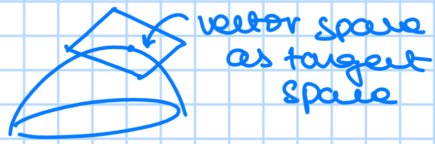
$= (bf - ce,\ dc - fa,\ ae - bd)$

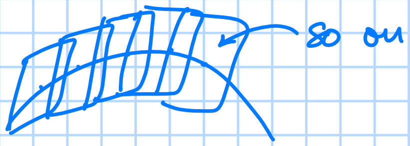
same as

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a & b & c \\ d & e & f \end{vmatrix} = (bf - ce,\ dc - fa,\ ae - bd)$$

27th march :

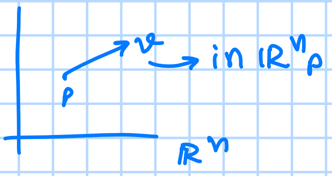
Tangent space :





Defn : (Tangent space) let $p \in \mathbb{R}^n$, we define the tangent space of $\mathbb{R}^n$ at p to be :

$$\mathbb{R}^n_p = \{ (p, v) \mid v \in \mathbb{R}^n \} \subseteq \mathbb{R}^n \times \mathbb{R}^n$$



($\mathbb{R}^n_p$ = all vectors where p is origin)

Think of this as a copy of $\mathbb{R}^n$ wrt p, if we define operations

$$+ : \mathbb{R}^n_p \times \mathbb{R}^n_p \longrightarrow \mathbb{R}^n_p$$
$$(p, v) + (p, w) := (p, v + w)$$
$$a.(p, v) := (p, a v)$$

then $\mathbb{R}^n_p$ becomes a vectorspace over $\mathbb{R}$

we will write (p, v) as v_p to denote a vector with basepoint p.
Call v_p as vector v at p.

Defn: (Usual inner product) $\langle \cdot, \cdot \rangle_p$ on $\mathbb{R}^n_p$

is $\langle v_p, w_p \rangle_p := \langle v, w \rangle$ (same by defination $\langle v_p, w_p \rangle_p = \langle v, w \rangle$)

Now, $\{ (e_1)_p, (e_2)_p, \ldots (e_n)_p \}$ is called usual basis for $\mathbb{R}^n_p$

Usual orientation of $\mathbb{R}^n_p$ is denoted by $(e_1)_p = (p, e_1)$

$$\mu = [(e_1)_p \ (e_2)_p \ . \ . \ . \ (e_n)_p]$$

vector field :

Defn : If A is an open set in $\mathbb{R}^n$, a vector field F on A is a function $F : A \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ s.t $F(p) \in \mathbb{R}^n_p \ \forall \ p \in A$ ($F : A \longrightarrow \mathbb{R}^n \times \mathbb{R}^n$ and $\mathbb{R}^n_p \subseteq \mathbb{R}^n \times \mathbb{R}^n$, $F(p) \in \mathbb{R}^n_p$)

for each p, we can write

$$F(p) = F^1(p)(e_1)_p + F^2(p)(e_2)_p + \cdots + F^n(p)(e_n)_p$$

and the n functions $F^i : A \longrightarrow \mathbb{R}$ are called the component functions of F.

as $(e_i)_p \in \mathbb{R}^n_p$

$F^i : A \longrightarrow \mathbb{R}$ $\quad F^i(p) \in \mathbb{R}$, $(e_i)_p \in \mathbb{R}^n_p$

Note : we will say vectorfield F is of class C^r if each $F^i : A \longrightarrow \mathbb{R}$ is of class C^r

$\left(F(p) = \sum_{i=1}^{n} F^i(p)(e_i)_p \right)$ ↳ component function

operations on vector fields :

If F, G are vectorfields on an open set A, and $f : A \longrightarrow \mathbb{R}$ is a function define

$$(F + G)(p) = F(p) + G(p) \qquad (F(p), G(p) \in \mathbb{R}^n_p)$$
$$\langle F, G \rangle_p = \langle F(p), G(p) \rangle$$
$$(f.F)(p) = f(p).F(p)$$

Differential forms:

Defn: A k-Tensor field on an open set $A \subseteq \mathbb{R}^n$ is a function T assigning to each $p \in A$, an element of $T^k(\mathbb{R}^n_p)$

(↙ k tensor field
$T(P) \in T^k(\mathbb{R}^n_p)$
$P \mapsto T^k(\mathbb{R}^n_p)$
$T(P) \in T^k(\mathbb{R}^n_p)$)

for $P \in A$ $T(P) \in T^k(\mathbb{R}^n_p)$

That is, $\forall p \in A$, $T(P)$ is a function which maps k-tuple of tangent vectors to $\mathbb{R}^n$ at P to $\mathbb{R}$

$$\forall P \in A,\ T(P) : \underbrace{\mathbb{R}^n_p \times \mathbb{R}^n_p \times \cdots \mathbb{R}^n_p}_{k \text{ times}} \longrightarrow \mathbb{R} \in T^k(\mathbb{R}^n_p)$$

$(T(P): \mathbb{R}^n_p \times \cdots \mathbb{R}^n_p \longrightarrow \mathbb{R} \in T^k(\mathbb{R}^n_p))$

It's value on a k-tuple of tangent vectors can be written as, $T(P)((v_1)_p, (v_2)_p, \ldots (v_k)_p)$

Defn: (Differential k-Form) $\forall P \in A$, $\omega(P) \in \Lambda^k(\mathbb{R}^n_p)$ for ω a k-Tensor field, then we say ω is a differential k-form on A

If $\{\varphi_1(P), \ldots \varphi_n(P)\}$ is a dual basis to $\{(e_1)_P, (e_2)_P, \ldots (e_n)_P\}$ for $\mathbb{R}^n_p$, then $\{\varphi_{i_1}(P) \wedge \varphi_{i_2}(P) \wedge \cdots \varphi_{i_k}(P) \mid 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$ is a basis for $\Lambda^k(\mathbb{R}^n_p)$, then (normal basis)

($\omega(P) \in \Lambda^k(\mathbb{R}^n_p)$ insted of $T^k(\mathbb{R}^n_p)$)

$$\omega(P) = \sum_{1 \leq i_1 < i_2 < \cdots i_k \leq n} \omega_{i_1, \ldots i_k}(P) \left(\varphi_{i_1}(P) \wedge \varphi_{i_2}(P) \wedge \cdots \varphi_{i_k}(P)\right)$$

($\omega_{i_1 i_2 \cdots i_k}(P) \in \mathbb{R}$ coeffs)

for some $\omega_{i_1, i_2, \ldots i_k} : A \longrightarrow \mathbb{R}$
↓
depends on P so $\omega_{i_1, \ldots i_k}(P) \in \mathbb{R}$
↓
some coefficient

Note: we say that ω is of class C^r if the functions $\omega_{i_1 \ldots i_k}$ are of class C^r.

Note: we can extend operations on alternating k-tensors to diffential k-forms

sum: $\omega + \eta$ → do at each P
product: $f \cdot \omega$ (where $f: A \longrightarrow \mathbb{R}$) → do at each P

wedge product: $\omega \wedge \eta$ → do at each p

Defn: If A is an open set in $\mathbb{R}^n$ and $f: A \longrightarrow \mathbb{R}$ a function, then f is called a scalar field
also f a diffential 0-field
(nothing as input and outputs a number)

Defn: For any $k \in \mathbb{Z}_{\geq 0}$ we write $\Omega^k(A)$ for the collection of all diffential k-forms of class C^∞ on A

$(f(P) \in \mathbb{R} = \Lambda^0(\mathbb{R}^n_p))$
↓
nothing as input

Exe: $\Omega^k(A)$ is a vector space

$$\Omega^k(A) = \left\{ \omega \;\middle|\; \begin{array}{l} \omega(P) \in \Lambda^k(\mathbb{R}^n_p) \\ \forall P \in A \;\& \\ \omega_{i_1, i_2 \ldots i_k} \in C^\infty \end{array} \right\}$$

$\omega_1 + \omega_2 \in \Omega^k(A)$ is trivial, $f\omega \in \Omega^k(A)$ is also trivial

Defn: (The differential of 0-form) let $A \subseteq \mathbb{R}^n$ be open, let $f : A \to \mathbb{R}$ be differentiable, we define a 1-form df on A by

$$df(P)(v_P) := Df(P)(v) \quad \left(df(P) \in \Lambda^1(\mathbb{R}^n_P) \right)$$

the 1-form df is called the differential of f

Some special 1-forms :

Recall the projection function $\pi^i : \mathbb{R}^n \longrightarrow \mathbb{R}$ $\left(f : A \to \mathbb{R} \text{ or } 0\text{-form} \right)$

$$x = (x^1, \dots, x^n) \longmapsto x^i$$

from these functions, using the differential we have 1-forms $d\pi^i$

for convinience we often abuse notation we write $d\pi^i = dx^i$

we use same symbol x^i for π^i

i.e

dx^i is the 1-form which satisfyes

$$\begin{aligned} dx^i(v_P) &:= d\pi^i(v_P) \\ &= D\pi^i(P)(v) \\ &= [0 \dots 1\ 0 \dots 0]\, v \\ &= v^i \end{aligned}$$

ith value

$\downarrow$

ith comp of v

$\left(\because \text{By defination } d\pi^i(v_P) = D\pi^i(P)(v) \right)$

so $\{dx^1(P), \dots dx^n(P)\}$ collection of 1-forms pick out v^i

so, they are just dual basis to $\{(e_1)_P, \dots (e_n)_P\}$

so, every k-form ω can be written as:

$$W(P) = \sum_{i_1 < i_2 < \dots i_k} \omega_{i_1 \dots i_k}(P)\ \underbrace{dx^{i_1}(P) \wedge dx^{i_2}(P) \wedge \dots dx^{i_k}(P)}_{\text{Basis element of } \Lambda^k(\mathbb{R}^n_P)}$$

Theorm: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then

$$\begin{aligned} df &= D_1 f\, dx^1 + \dots D_n f\, dx^n \\ &= \frac{\partial f}{\partial x^1} dx^1 + \dots \frac{\partial f}{\partial x^n} dx^n \end{aligned}$$

proof:

$$\begin{aligned} df(v_P) &= Df(P)(v) \\ &= \sum_{i=1}^n D_i f(P)(v)^i \\ &= \sum_{i=1}^n D_i f(P)\, dx^i(P)(v_P) \end{aligned}$$

$$\Rightarrow df = \sum_{i=1}^n D_i f\, dx^i$$

$$\left(\because df(v_P) = \sum_{i=1}^n D_i f(P)\, dx^i(P)(v_P) \right)$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$

$Df : [D_1 f \dots D_n f]_{1 \times n}$

$$Df(P)(v) = [D_1 f(P) \dots D_n f(P)] \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} = \sum_{i=1}^n D_i f(P)\, v^i$$

1st Apr :

Recap : tangent space to $\mathbb{R}^n$ at p denoted by $\mathbb{R}^n_p$

vector field $F(P) = F^1(P)(e_1)_p + \cdots + F^n(P)(e_n)_p$

tensor fields of alt tensors or differential forms/differential k-forms

dx^i of the 1form which satisfies $dx^i(P)(v_p) = v^i$

(diff 1-form) $dx^i(P) : \mathbb{R}^n_p \longrightarrow \mathbb{R}$

$v_p \longrightarrow v^i$

$dx^i(P) \in \Lambda^1(\mathbb{R}^n_p)$

every k-form ω can be written as $dx^i(P)(v_p) = v^i$

$$\omega = \sum_{i_1 < i_2 \cdots i_k} \omega_{i_1 i_2 \ldots i_k} \, dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_k}$$

$\omega_{i_1, i_2, \ldots i_k}$ for some funtions

Eg : 1-form on $\mathbb{R}^3 = \{(x, y, z)\}$

$\omega = xz\,dx + z^2 dy$

$\theta = dy + x\,dz$

now $\omega \wedge \theta = (xz\,dx + z^2 dy) \wedge (dy + x\,dz)$

$= xz\,dx \wedge dy + \cancel{z^2 dy \wedge dy} + x^2 z\,dx \wedge dz + xz^2 dy \wedge dz$

$= xz\,dx \wedge dy + x^2 z\,dx \wedge dz + xz^2 dy \wedge dz$

$\omega(x, y, z) \in \Lambda^1(\mathbb{R}^3_{(x,y,z)})$

Eg : 2-form on $\mathbb{R}^4$:

$\eta = dx^1 \wedge dx^4 - \cos(x^1 + x^2 + x^3)\, dx^3 \wedge dx^4$

is a 2-form on $\mathbb{R}^4$

The differential of 0-form :

$A \subseteq \mathbb{R}^n$ and $f : A \longrightarrow \mathbb{R}$ is differentiable then a 1form on A

$$df(P)(v_p) = Df(P)(v)$$

$$\Rightarrow df = \sum_{i=1}^{n} D_i f \, dx^i \quad \left(\text{here } df = \sum \omega_i \, dx^i, \ \omega_i = D_i f \right)$$

Eg : $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$

$f(x, y, z) = xy^2 z^3$

$$df = \begin{bmatrix} y^2 z^3 & 2xyz^3 & 3xy^2 z^2 \end{bmatrix} \begin{bmatrix} dx \\ dy \\ dz \end{bmatrix}$$

$$= y^2 z^3 dx + 2xyz^3 dy + 3xy^2 z^2 dz$$

Eg : $f : \mathbb{R}^4 \longrightarrow \mathbb{R}$

$f(x^1, x^2, \ldots x^4) = \sin(x^1 x^2 x^3 x^4)$

then

$$df = \cos(x^1 x^2 x^3 x^4) \left[\sum_{i_1 \neq i_2 \neq i_3 \neq i_4} x^{i_1} x^{i_2} x^{i_3} dx^{i_4} \right]$$

Defn : (Divergence) F is a vector field on $A \overset{\text{open}}{\subseteq} \mathbb{R}^n$, we define div of F denoted by divF as:

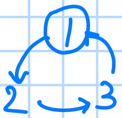
$$\operatorname{div} F = \sum_{i=1}^{n} D_i F$$

$\left(\begin{array}{l} F(x, y, z) = (xy^2, 2xz, 4z^2) \\ \operatorname{div} F = y^2 + 0 + 8z = y^2 + 8z \end{array} \right)$

(F is a vector field, so $F(P) \in \mathbb{R}^n_p$ & $F(P) = F^1(P)(e_1)_p + \ldots F^n(P)(e_n)_p$)

Defn: (curl) If F is a vector field on $A \underset{\text{open}}{\subseteq} \mathbb{R}^3$ we define curl of F:

$$\nabla \times F = (D_2F^3 - D_3F^2)(e_1)_p$$
$$+ (D_3F^1 - D_1F^3)(e_2)_p$$
$$+ (D_1F^2 - D_2F^1)(e_3)_p$$



$(D_2F^3 - D_3F^2)$

$g: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$Dg = \begin{bmatrix} D_1 g \cdots D_n \end{bmatrix}_{m \times n}$$

pushback, pullforward:

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is diff funtion $Df(p)$

$Df(p): \mathbb{R}^n \longrightarrow \mathbb{R}^m$

we can view $Df(p)$ as $\mathbb{R}^n_p$ to $\mathbb{R}^m_{f(p)}$ by $f_*: \mathbb{R}^n_p \longrightarrow \mathbb{R}^m_{f(p)}$ defined by

$$\left(f_*: \mathbb{R}^n_p \longrightarrow \mathbb{R}^m_{f(p)}\right)$$
$$v_p \longmapsto (Df(p)(v))_{f(p)}$$

$$f_*(v_p) = (Df(p)(v))_{f(p)}$$

we say sometimes $(Df(p)(v))_{f(p)}$ is push forward of v_p under f

The linear trasformation: $f_*: \mathbb{R}^n_p \longrightarrow \mathbb{R}^m_{f(p)}$

induces a linear trasformation from $\Lambda^k(\mathbb{R}^m_{f(p)})$ to $\Lambda(\mathbb{R}^n_p)$

$$f^*: \Lambda^k(\mathbb{R}^m_{f(p)}) \longrightarrow \Lambda^k(\mathbb{R}^n_p)$$

$$\Big(f^*: \underset{\substack{\downarrow \\ \omega \text{ form on } \mathbb{R}^m}}{\Lambda^k(\mathbb{R}^m_{f(p)})} \rightarrow \underset{\substack{\downarrow \\ \omega \text{ form on } \mathbb{R}^n}}{\Lambda^k(\mathbb{R}^n_p)}\Big)$$

so for ω a k-form of $\mathbb{R}^m$ we get

$f^*\omega$ a k form on $\mathbb{R}^n$

called the pull back of ω under f

$$(f^*\omega)(p) = f^*(\omega(f(p)))$$

$v_1, v_2 \dots v_k \in \mathbb{R}^n_p$ then

$$(f^*\omega)(p)(v_1, \dots v_k) = \underbrace{\omega(f(p))}_{\in \Lambda^k(\mathbb{R}^m_{f(p)})}(\overset{\in \mathbb{R}^m_{f(p)}}{f_*(v_1)}, \overset{\in \mathbb{R}^m_{f(p)}}{f_*(v_2)}, \overset{\dots}{\dots} \overset{\in \mathbb{R}^m_{f(p)}}{f_*(v_k)})$$

Theorm: (properties of f^*) If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable then for $i = 1 \dots m$

① $$\underset{\substack{\uparrow \\ \text{1-form on } \mathbb{R}^m}}{f^*(dx^i)} = \sum_{j=1}^{n} D_j f^i . dx^j \leftarrow \text{1-form on } \mathbb{R}^n$$

$$= \sum_{j=1}^{n} \frac{\partial f^i}{\partial x^j} dx^j$$

② $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$

③ $f^*(g.\omega) = (g \circ f) f^*(\omega)$

④ $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$

proof:

① $$f^*(dx^i)(p)(v_p) = dx^i(f(p))\big(f_*(v_p)\big) \quad (\text{By defination})$$

$$= dx^i(f(p))\,(Df(p)(v))_{f(p))}$$

$$= dx^i(f(p))\left(\sum_{j=1}^{n} D_j f^1(p) v^j, \dots \sum_{j=1}^{n} D_j f^m(p) v^j\right)_{f(p)}$$

$$= \sum_{j=1}^{n} D_j f^i(p) v^j \quad \text{(Property of } dx^i(f(p))\text{)}$$

$$= \sum_{j=1}^{n} D_j f^i(p)\, dx^j(p)(v_p)$$

③ $(f^*(g\cdot\omega))(p)(v_1 \dots v_k)$

$$= (g\cdot\omega)(f(p))(f_*v_1, \dots f_*v_k)$$

$$= g(f(p))\cdot \underbrace{\omega(f(p))(f_*v_1 \dots f_*v_k)}_{(f^*\omega)(p)(v_1 \dots v_k)}$$

$$\Rightarrow \quad f^*(g\cdot\omega) = (g\circ f)\, f^*(\omega)$$

② $f^*(\omega_1+\omega_2)(p)(v_1, \dots v_k) = (\omega_1+\omega_2)(f(p))(f_*(v_1) \dots f_*(v_k))$

$$= \omega_1(f(p))(f_*(v_1), \dots f_*(v_k)) + \omega_2(f(p))(f_*(v_1), \dots f_*(v_k))$$

$$= f^*(\omega_1) + f^*(\omega_k)$$

④ $f^*(\omega\wedge\eta) = f^*(\omega\wedge\eta)(p)(v_1 \dots v_k) = (\omega\wedge\eta)(f(p))(f_*(v_1) \dots f_*(v_k))$

$$= \omega(f(p))\wedge\eta(f(p))(f_*(v_1) \dots f_*(v_k))$$

$$= f^*(\omega)\wedge f^*(\eta)$$

Eg: $f^*(P\,dx^1\wedge dx^2 + Q\,dx^2\wedge dx^3)$

$$= (P\circ f)\,[f^*(dx^1)\wedge f^*(dx^2)] + (Q\circ f)\,[f^*(dx^2)\wedge f^*(dx^3)]$$

Theorm: If $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is differentiable, then

$$f^*(\underbrace{h\,dx^1\wedge \dots \wedge dx^n}_{\Lambda^n(\mathbb{R}^n_{f(p)})}) = (h\circ f)(\det f')(\underbrace{dx^1\wedge \dots \wedge dx^n}_{\Lambda^n(\mathbb{R}^n_p)})$$

proof: By previous theorm

$$f^*(h\,dx^1\wedge\dots\wedge dx^n) = (h\circ f)\, f^*(dx^1\wedge\dots\wedge dx^n)$$

so it is enough to show that

$$f^*(dx^1\wedge\dots\wedge dx^n) = (\det f')(dx^1\wedge dx^2 \dots\wedge dx^n)$$

let $p\in\mathbb{R}^n$ $f'(p) = A = (a_{ij})$

then

$$f^*(dx^1\wedge\dots dx^n)(p)((e_1)_p, \dots (e_n)_p)$$

$$= (dx^1\wedge\dots dx^n)\, f(p)\left(f_*(e_1)_p, f_*(e_2)_p, \dots f_*(e_n)_p\right)$$

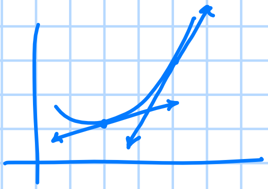
$$= (dx^1\wedge\dots dx^n)(f(p))\left(\sum_{i=1}^{n} a_{i1}(e_i)_{f(p)}, \dots \sum_{i=1}^{n} a_{in}(e_i)_{f(p)}\right)$$

(By theorm from last week) (By transforming it)

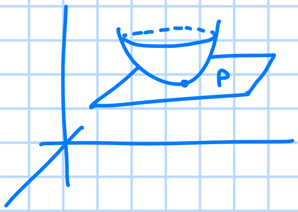
$$= (\det a_{ij})(dx^1\wedge\dots dx^n)(f(p))((e_1)_{f(p)}, \dots (e_n)_{f(p)})$$

3rd Apr :

Reason for $(\theta)_p$:

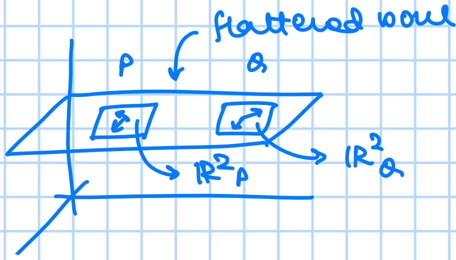


at each point we have a tangent (copy of $\mathbb{R}$)
Similarly for 3D, at each point we have a tangent plane (copy of $\mathbb{R}^2$)



our notation: $\mathbb{R}^n_p$ is the space $\mathbb{R}^n$, taget plane at p
intuitively make the bowl flatter and flatter so the bowl goes to $\mathbb{R}^2$ copy sitting inside $\mathbb{R}^3$

so, $\mathbb{R}^2_P$, $\mathbb{R}^2_Q$ are different vector space, even tho they seem coinciding, the vectors in $\mathbb{R}^2_P$ & $\mathbb{R}^2_Q$ are different.



F ← vector field

then

$F(P) \in \mathbb{R}^n_P$ so $F(P) = F^1(P)(e_1)_P + F^2(P)(e_2)_P$

Recap: Example computing df for $f: A \to \mathbb{R}$, df a 1-form given by

$$df = \sum_{\alpha=1}^{n} D_\alpha f \, dx^\alpha$$

$\left(f: \mathbb{R}^n \longrightarrow \mathbb{R} \quad df = [\quad]_{1\times n} [\quad]_{n\times 1} \right)$

pullback and pushforward:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^m \\
\mathbb{R}^n_p \arrow[r, "f_*"] & \mathbb{R}^m_{f(p)} \\
\Lambda^k(\mathbb{R}^n_p) & \Lambda^k(\mathbb{R}^n_{f(p)}) \arrow[l, "f^*"']
\end{tikzcd}$$

$\left(\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m,\ \mathbb{R}^n_p \xrightarrow{f_*} \mathbb{R}^m_{f(p)} \quad \Lambda^k(\mathbb{R}^m_{f(p)}) \xrightarrow{f^*} \Lambda^k(\mathbb{R}^n_p) \right)$

The differential (of a k-form):

For $A \subseteq \mathbb{R}^n$ open, $\omega \in \Omega^k(A)$ ← all differential k-forms on A
$\omega(P) \in \Lambda^k(\mathbb{R}^n_P)$

$$\omega = \sum \omega_{i_1 \dots i_k} \, dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_k} \quad \text{(By defination)}$$

Defn: ($k+1$ form $d\omega$ or differential of ω) we denote $d\omega$

$$d\omega = \sum d\omega_{i_1 \dots i_k} \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$

as $\omega_{i_1 \dots i_k} : A \xrightarrow{i_1 < \dots < i_k} \mathbb{R}$ ← making them 1-form

$$d\omega_{i_1 \dots i_k} = \sum_{\alpha=1}^{n} D_\alpha \omega_{i_1 \dots i_k} dx^\alpha$$

so, $$d\omega = \sum_{i_1 < \dots < i_k} \Big[\underbrace{\sum_{\alpha=1}^{n} (D_\alpha \omega_{i_1 \dots i_k} dx^\alpha}_{1\text{-form}} \Big] \wedge dx^{i_1} \wedge dx^{i_2} \dots \wedge dx^{i_k}$$

In other words, we have operator d: ← differential k+1 form (C^∞)

$$d: \Omega^k(A) \longrightarrow \Omega^{k+1}(A)$$

eg: $\eta = z\,dx + xy\,dy$ is 1-form on $\mathbb{R}^3$

$$d\eta = d(z) \wedge dx + d(xy) \wedge dy$$

$$= \left[\frac{\partial(z)}{dx} dx + \frac{d(z)}{dy} dy + \frac{d(z)}{dz} dz \right] \wedge dx + \left[\frac{\partial}{\partial x}(xy)\, dx + \frac{d(xy)}{dy} dy + \frac{d(xy)}{d(z)}, d(z) \right] \wedge dy$$

$$= dz \wedge dx + y\,dx \wedge dy + \underbrace{x\,dy \wedge dy}_{0}$$

$$= dz \wedge dx + y\,dx \wedge dy$$

eg: $d(dx^{i_1} \wedge \dots \wedge dx^{i_k})$
$$= \sum_{\alpha=1} (D_\alpha(1)\,dx^\alpha) \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}$$
$$= 0$$

Theorm : (I) $d(\omega + \eta) = d\omega + d\eta$

(II) If ω is a k-form and η is an l-form then

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$$ (compare it with product rule)

(III) $d(d\omega) = 0$

(IV) If ω is a k-form on $\mathbb{R}^m$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable then

$$f^*(d\omega) = d(f^*\omega)$$

proof : (I) as we are diffentiating the fuction, from defination

$$d(\omega + \kappa) = \sum \left(\sum D_\alpha (\omega_{i_1 \dots i_k} + \kappa_{i_1 \dots} dx^\alpha) \right) \wedge dx^{i_1} \wedge dx^{i_2} \dots dx^{i_k}$$
$$= \sum \left[\sum D_\alpha (\omega_{i_1 \dots} dx^{k\alpha}) \right] \wedge dx^{i_1} \wedge dx^{i_2} \dots dx^{i_k} + \sum\sum D_\alpha \kappa_{x_i \dots} dx^{i_k}$$
$$= d(\omega) + d(\kappa)$$

(II) Case I : ω is a 0-form f, then
$$d(\omega \wedge \eta) = d(f\eta)$$
$$= d\Big(\sum_{i_1 < \dots < i_l} f\eta_{i_1 \dots i_l}\, dx^{i_1} \wedge \dots dx^{i_l}\Big)$$

so by defn: $$d(\omega \wedge \eta) = \sum_{i_1 < \dots < i_l} \sum_{\alpha=1}^{n} D_\alpha (f \eta_{i_1 \dots i_l})\, dx^\alpha \wedge dx^{i_1} \wedge \dots \wedge dx^{i_l}$$

$$= \sum_{i_1 < \dots < i_l} \Big[\sum_{\alpha=1}^{n} (D_\alpha f)\eta_{i_1 \dots i_l} + f(D_\alpha \eta_{i_1 \dots i_l}) \Big] dx^\alpha \wedge dx^{i_1} \dots \wedge dx^{i_l}$$

$$= df \wedge \eta + (-1)^0 f \wedge d\eta$$

so true for case I

case II : $\omega = dx^{i_1} \wedge \dots dx^{i_k}$
$\eta = dx^{j_1} \wedge \dots dx^{j_l}$
then $d\omega = 0$ (seen in enaple)
$d\eta = 0$
now $\omega \wedge \eta = dx^{i_1} \wedge \dots dx^{i_k} \wedge dx^{j_1} \dots \wedge dx^{j_l}$
$d(\omega \wedge \eta) = 0$
so $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

case III : general case α is n-form, ω is m form, I, K sum of all basic rows

$$d(\alpha \wedge \omega) = d\Big(\sum \alpha_I\, dx^I \wedge \sum \omega_K\, dx^K\Big)$$
$$= d\Big(\sum\sum \alpha_I \omega_K\, dx^I \wedge dx^K\Big)$$
$$= \sum\Big(\sum\Big(\sum_i \partial_i(\alpha_I)\omega_K (dx_i \wedge dx^I) \wedge dx^K + \alpha_I\, \partial_i(\omega_K)\, dx_i \wedge (-1)^{mn} dx^K \wedge dx^I\Big)\Big)$$

$$= \sum\Big(\sum\Big(\sum_i \partial_i(\alpha_I)\omega_K (dx^i \wedge dx^I) \wedge dx^K$$
$$+ \alpha_I(\partial_i \omega_K)(-1)^{mn}(-1)^{mn}(-1)^n dx^I \wedge dx^i \wedge dx^K\Big)\Big)$$

$$= \sum\sum (d(\alpha_I)\omega_K\, dx^I \wedge dx^K + (-1)^n \alpha_I\, d(\omega_K)\, dx^I \wedge dx^K)$$

$$= d\alpha \wedge \omega + (-1)^n \alpha \wedge d\omega$$

(III) lets claim $d(d\omega) = 0$

now

$$d\omega = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} D_\alpha \omega_{i_1 \ldots i_k} \, dx^\alpha \wedge dx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_k}$$

$$d(d\omega) = \sum_{i_1 < \cdots < i_k} \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} D_{\alpha,\beta} \omega_{i_1 \ldots i_k} \, dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

here, $D_{\alpha,\beta}\, w_{i_1 \ldots i_k} \, dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \wedge \ldots dx^{i_k}$

$$= D_{\beta,\alpha}\, w_{i_1 \ldots i_k} \, dx^\beta \wedge dx^\alpha \wedge \ldots\, dx^{i_k}$$

$$= D_{\beta,\alpha}\, w_{i_1 \ldots i_k} (-1) \, dx^\alpha \wedge dx^\beta \wedge \ldots dx^{i_k}$$

$$\Rightarrow D_{\alpha,\beta} w_{i_1 \ldots i_k} \, dx^\beta \wedge dx^\alpha \wedge dx^{i_1} \ldots \wedge dx^{i_k} + D_{\beta,\alpha} \omega_{i \ldots i_k} \, dx^\beta \underset{=0}{\wedge} dx^\alpha \ldots \wedge dx^{i_k}$$

so by symmetry $d(d\omega) = 0$

(IV) we claim that if $\omega \in \Omega^k(\mathbb{R}^m)$ and $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable then $\underset{k+1 \text{ form}}{f^*(d\omega)} = \underset{k+1 \text{ form}}{d(f^*\omega)}$

<u>case I</u> : if ω is a 0-form on $\mathbb{R}^m$, i.e ω is a function of $g : \mathbb{R}^m \to \mathbb{R}$

$$f^*(d\omega) = f^*\left(\sum_{\alpha=1}^{m} D_\alpha g \, dx^\alpha\right)$$

$$= \sum_{\alpha=1}^{m} D_\alpha g \, f^*(dx^\alpha)$$

$$= \sum_{\alpha=1}^{m} (D_\alpha g) \sum_{j=1}^{n} \frac{\partial f^\alpha}{dx^j} dx^j$$

$$= \sum_{\alpha=1}^{m} \sum_{j=1}^{n} (D_\alpha g) \frac{df^\alpha}{dx^j} dx^j$$

and

$f^*\omega = g \circ f$, so

$$d(f^*\omega) = d(g \circ f) = \sum_{j=1}^{n} (D_j(g \circ f)) \, dx^j$$

$$= \sum_{j=1}^{n} \left(\sum_{\alpha=1}^{m} D_\alpha g \, \frac{\partial f^\alpha}{dx^j} \right) dx^j$$

$$= f^*(d\omega)$$

<u>case II</u> : general case:

By induction (IV) is true for ω is a k-form then for k+1 form

also it is enough to prove it for $\omega \wedge dx^i$ (this is due to basis)

so we have to show:

$$f^*(d(\omega \wedge dx^i)) = d f^*(\omega \wedge dx^i)$$

now, $f^*(d(\omega \wedge dx^i)) = f^*(d\omega \wedge dx^i + (-1)^k \omega \wedge d(dx^i))$

$$= f^*(d\omega \wedge dx^i)$$

$$= f^*(d\omega) \wedge f^*(dx^i)$$

$$= d(f^*\omega) \wedge f^*(dx^i) + 0 \quad \text{(By induction Hypothesis)}$$

$$= d(f^*\omega) \wedge f^*(dx^i) + (-1)^k f^*\omega \wedge \underbrace{d(f^* dx^i)}_{= f^*(d(dx^i)) \text{ By induction hypothesis } = 0}$$

$$= d(f^*\omega \wedge f^* dx^i)$$

$= d(f^*(\omega \wedge dx^i))$

so $f^*(d\omega) = d(f^*\omega)$

f^* is a pullback

7^{th} Apr:

Recap: $d: \Omega^k(A) \longrightarrow \Omega^{k+1}(A)$ ($\Omega^k(A)$ is set of all C^∞ diff k forms on A)

Closed form, Exact form:

defn: let A be an open set in $\mathbb{R}^n$
a k-form ω on A with $k \geq 0$ is closed if $d\omega = 0$

Eg: $\omega = x\,dy + y\,dx$ on $\mathbb{R}^2$

$$d\omega = dx \wedge dy + dy \wedge dx$$
$$= dx \wedge dy - dx \wedge dy$$
$$= 0$$

defn: A 0-form on A is said to be exact on A if it is constant, on A.
A k-form $\omega \in \Omega^k(A)$, $k \geq 1$ is said to be exact if
$\exists (k-1)$ form η on A s.t
$\omega = d\eta$

Note: The theorm from past gives $d^2 = 0$ for any differtial form

so if ω is exact, i.e
$\omega = d\eta$
$\Rightarrow d\omega = d(d\eta) = 0$
so every exact form is closed

Eg:

$\omega = x\,dx$ is an exact 1-form on $\mathbb{R}^1$ as
$\eta = \frac{x^2}{2}$

then $d\eta = d\left(\frac{x^2}{2}\right) = \frac{d}{dx}\left(\frac{x^2}{2}\right)dx$
$= x\,dx$

In $\mathbb{R}^3$, a vector field:

$$F = F^1\hat{i} + F^2\hat{j} + F^3\hat{k}$$
$$(F(P) = F^1(P)(\varepsilon_1)_P + F^2(P)(\varepsilon_2)_P + F^3(P)(\varepsilon_3)_P)$$

Defn: F is said to be conservative if it is the gradient of some scalar field f

i.e $F^1 = \frac{df}{dx}$
$F^2 = \frac{df}{dy}$
$F^3 = \frac{df}{dz}$

this is same as saying 1-form

$\omega = F^1dx + F^2dy + F^3dz$ is exact

i.e $\omega = d\underset{\text{0-form}}{f}$

$\omega = df$

$$F^1dx + F^2dy + F^3dz = \frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz$$

so for a vector field if $\omega = F^1dx + F^2dy + F^3dz$ is exact, vector field is conservative

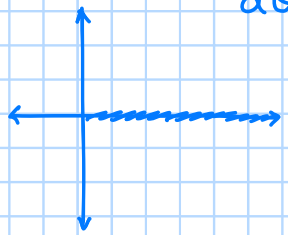
Eg: 1-form on $\mathbb{R}^2 \setminus \{0\}$

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

then $d\omega = 0$

let θ be defined uniquely on $(0, 2\pi)$
then for

$\cos\theta = \frac{x}{\sqrt{x^2+y^2}}$ $\sin\theta = \frac{y}{\sqrt{x^2+y^2}}$



$d\theta = \omega$ on $\{(x,y) \mid x<0, \text{ or } x \geq 0 \text{ and } y \neq 0\}$
note that θ cannot be defined continiously on all of $\mathbb{R}^2 \setminus \{0\}$

I claim ω is <u>not</u> exact
if $\omega = df$ for $f : \mathbb{R}^2\setminus\{0\} \longrightarrow \mathbb{R}$

$\left.\begin{matrix} D_1 f = D_1\theta \\ D_2 f = D_2 \end{matrix}\right\}$ as only one line, to make f cont $df = d\theta$

and so $f = \theta + \text{const}$ and so such f does not exist

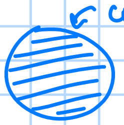
Note: From above example we get closed form $\not\Rightarrow$ exact
but exact $\Rightarrow$ closed
we want to now know when is a closed form exact.

Defn: let $A \subseteq \mathbb{R}^n$ open, we say A is star shaped (or star convex) w.r.t p of A if $\forall x \in A$, the line segment joining x and p lies in A

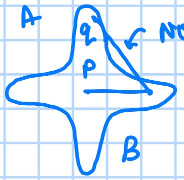
Recall that U is convex if $\forall x, y \in U$, $\alpha x + (1-\alpha) y \in U$ $\forall \alpha \in [0,1]$

Eg:

convex set



also star-shaped set wrt every point $p \in A$



Not convex

B is star-shaped wrt to p but not wrt q

C is not-star shaped wrt any point

Theorm: (Poincare lemma) let A be a star-shaped open set in $\mathbb{R}^n$
If ω is a closed k-form on A then ω is exact.

Note: let A be a set which is star-shaped wrt $p \in A$
then for $\omega \in \Lambda^k(A)$
s.t $d\omega = 0$
$\Rightarrow \omega$ is exact
or $\exists \eta \in \Lambda^{k-1}(A)$ s.t
$\omega = d\eta$

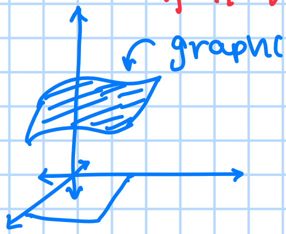
(closed $\Rightarrow$ exact if A is starshaped)

Geometric preliminaries:

Cubes and chains:

Defn: The set $[0,1]^k = \underbrace{[0,1] \times \cdots [0,1]}_{k \text{ times}}$ in $\mathbb{R}^k$ is called the standard k-cube in $\mathbb{R}^k$ for $k \geq 1$.

if $k=0$ then $\mathbb{R}^0$ and $[0,1]^0$ both denote $\{0\}$ (singleton 0)

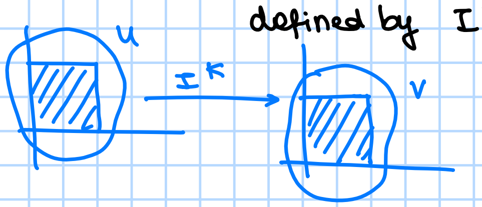


Defn: let $U \subseteq \mathbb{R}^k$ be open, containing standard cube $[0,1]^k$
let $V \subseteq \mathbb{R}^n$ be open & let $c : U \longrightarrow V$ be C^∞
we say $c : [0,1]^k \longrightarrow V$ is a k-cube of class C^∞ (or k-cube) in V

we will think of the standard k-cube in $\mathbb{R}^k$ as k-cube of class C^∞, given by the function

$$I^k : [0,1]^k \longrightarrow \mathbb{R}^k$$

defined by $I^k(x) = x$, $\forall\, x \in U$



(I^k is the k-cube in $\mathbb{R}^k$)

Defn: An expression consisting of a finite sum of k-cubes in $V \subseteq \mathbb{R}^n$ with integer coefficients is called k-chain in V

Eg: c_1, c_2, c_3 are k-cubes in V then ($c_i : [0,1]^k \longrightarrow V$ and is C^∞)
$c = 2c_1 - 3c_2 + 5c_3$ is an example of a k-chain in V

Eg: If c_1 is a k-cube V, we can think of it as a k-chain $1 \cdot c_1$

k-chains can be added and multiplied by integers

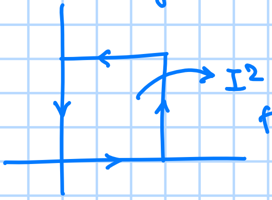
Eg: $2(c_1 + 3c_4) + (-2)(c_1 + c_2 + c_3)$
$= -2c_2 - 2c_3 + 6c_4$

The boundary of a k-chain:

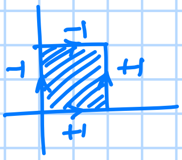
Defn: For each k-chain c in $V \subseteq \mathbb{R}^n$ we define (k-1) chain in V called the boundary of c, denote it by ∂c.

Let's start by defining boundary of standard k-cube I^k

Eg: Boundary of I^2 can be defined as sum of four 1-cubes arranged anti-clockwise around boundary of $[0,1]^2$

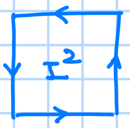


∂I^2 is the sum (with coeff given in picture) of the four 1-cubes as indicated.



8th April:

Recap: closed form and exact form ($d\omega=0$ or $\omega=d\eta$)
poincare lemma (when does $d\omega=0 \Rightarrow \exists \eta$ s.t $\omega=d\eta$)
geometric preliminaries:
K-cube, K-chain $\quad\quad$ ↙ A chain
$c:[0,1]^K \longrightarrow \mathbb{R}^n \quad\quad 2c_1-3c_2+7c_3$ (integer multiples)

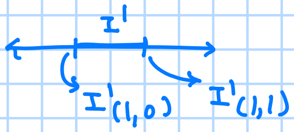


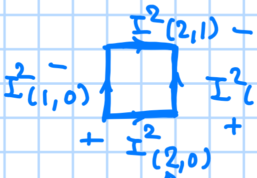
$I^2:[0,1]^2 \longrightarrow \mathbb{R}^2$ (Standard 2-cube)
s.t $\quad I^2(x)=x \;\forall x\in[0,1]^2$

now to preciselly define boundary of I^K (denoted by ∂I^K) in general, we first need these definations:

defn: $\forall i$ s.t $1\leq i\leq K$, we define $(K-1)$ cube $I^K_{(i,0)}$ and $I^K_{(i,1)}$ as follows:
If $x\in[0,1]^{K-1}$
then $I^K_{(i,0)} = I^K(x^1,\dots x^{i-1},0,x^i,\dots\; x^{K-1})$ $\quad$ ($I^K_{(i,\gamma)}$ is K-1 cube)
$I^K_{(i,1)} = I^K(x^1,\dots x^{i-1},1,x^i,\dots\; x^{K-1})$
1 at ith position
we call $I^K_{(i,0)}$ the $(i,0)$-face of I^K and
$I^K_{(i,1)}$ as the $(i,1)$-face of I^K

Eg:





$I^2_{(1,1)} = I^2(1,x)$ $\quad$ x always 1
$x\in[0,1]$

0 at second position

$\delta I^2 = I^2_{(2,0)} - I^2_{(2,1)} + I^2_{(1,1)} - I^2_{(1,0)}$

Defn: Boundary of I^K (denoted by ∂I^K):

$$\partial I^K = \sum_{i=0}^{K}\sum_{\alpha=0,1}(-1)^{i+\alpha} I^K_{(i,\alpha)}$$

(this is a K-1 chain)

$$\left(\delta I^K = \sum_{i=0}^{K}\sum(-1)^{i+\alpha} I^K_{(i,\alpha)}\right)$$

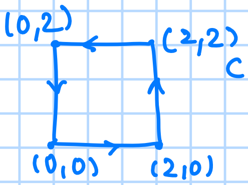
Defn: For general K-cube
$c:[0,1]^K \longrightarrow V\subseteq \mathbb{R}^n$
we define (i,α)-face
$c_{(i,\alpha)} = c\circ I^K_{(i,\alpha)}$ $\quad$ ↙ K-1 chain
then boundary $\quad \delta c = \sum_{i=1}^{K}\sum_{\alpha=0,1}(-1)^{i+\alpha}c_{(i,\alpha)}$ $\quad$ (same but with $c_{(i,\alpha)}$)

Now we have defined δc (Boundary of c) then now for a k-chain

Defn: $\sum_{i=1}^{m} a_ic_i$, a k-chain, we define it's Boundary as:

$$\partial\left(\sum_{i=1}^{m}a_ic_i\right) = \sum_{i=1}^{m}a_i\,\partial(c_i)$$

Eg:



$c:[0,1]^2 \longrightarrow \mathbb{R}^2$
$x \longrightarrow 2x$
so $c = 2\circ I^2$

$\sum a_ic_i$ $\quad$ ↙ k-chain
$\Rightarrow \delta(\sum a_ic_i)$
$= \sum a_i(\delta c_i)$
↙
k-1 chain

Theorm: If c is a k-chain in V, then $\partial(\partial c) = 0$, briefly $\partial^2 = 0$

(Boundary of a boundary = 0)

Proof: let $i \leq j$, consider

$(I^k_{(i,\alpha)})_{(j,\beta)}$ → this is a k−2 cube

If $x \in [0,1]^{k-2}$ then

$$(I^k_{(i,\alpha)})_{(j,\beta)}(x) = I^k_{(i,\alpha)}(I^{k-1}_{(j,\beta)}(x))$$
$$= I^k_{(i,\alpha)}(x^1, \dots x^{j-1}, \beta, x^j, \dots x^{k-2})$$
$$= I^k(x^1, \dots x^{i-1}, \alpha, x^i, \dots x^{j-1}, \beta, x^j, \dots x^{k-2})$$

now $(I^k_{(j+1,\beta)})_{(i,\alpha)}(x) = I^k_{(j+1,\beta)}(I^{k-1}_{(i,\alpha)}(x))$

$$= I^k_{(j+1,\beta)}(x^1, \dots x^{i-1}, \alpha, x^i, \dots x^{k-2})$$
$$= I^k(x^1, \dots x^{i-1}, \alpha, \dots x^i, \dots x^{j-1}, \beta, x^j, \dots x^{k-2})$$

so, $(I^k_{(i,\alpha)})_{(j,\beta)} = (I^k_{(j+1,\beta)})_{(i,\alpha)}$ for $i \leq j$

⇒ for any cube: $i \leq j$

$$(c_{(i,\alpha)})_{(j,\beta)} = (c_{(j+1,\beta)})_{(i,\alpha)}$$

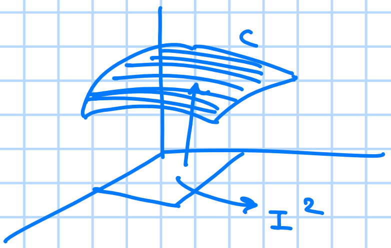
(we show $(I^k_{(i,\alpha)})_{(j,\beta)} = (I^k_{(j+1,\beta)})_{(i,\alpha)}$) But sign differ ⇒ 0

now, $\partial(\partial c) = \partial\left(\sum_{i=1}^{k}\sum_{\alpha=0,1}(-1)^{i+\alpha}c_{(i,\alpha)}\right)$

$$= \sum_{j=1}^{k-1}\sum_{\beta=0,1}(-1)^{j+\beta}\left(\sum_{i=1}^{k}\sum_{\alpha=0,1}(-1)^{i+\alpha}c_{(i,\alpha)}\right)_{(j,\beta)}$$
$$= \sum_{j=1}^{k-1}\sum_{\beta=0,1}\sum_{i=1}^{k}\sum_{\alpha=0,1}(-1)^{i+\alpha+j+\beta}(c_{(i,\alpha)})_{(j,\beta)}$$

where $(-1)^{i+\alpha+j+\beta}(c_{(i,\alpha)})_{(j,\beta)}$ and $(-1)^{i+j+1+\alpha+\beta}(c_{(j+1,\beta)})_{(i,\alpha)}$ appear with opposite sign, so all terms cancle and $\partial(\partial c) = 0$

Note: The above theorm is for any k-cube c. This follows that it is true for any k-chain



projecting $I^2 \to c$
$c \to I^2$

Integrating forms over chains:

Defn: If ω is a k-form on an open set U containing $[0,1]^k$, we can write

$$\omega = f\,dx^1 \wedge dx^2 \wedge dx^3 \dots \wedge dx^k$$ for some function f

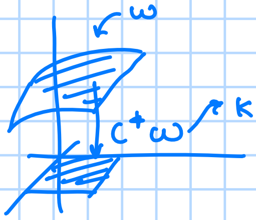
we define $\int_{[0,1]^k} \omega := \int_{[0,1]^k} f$ $\quad (\omega \in \Omega^k(A),\ A \underset{\text{open}}{\supseteq} [0,1]^k)$

we can also write it as

$$\int_{[0,1]^k} f\, dx^1 \wedge dx^2 \ldots \wedge dx^k = \int_{[0,1]^k} f(x^1, \ldots x^k)\, dx^1 \ldots dx^k$$

Defn : If ω is a k-form on $V \subseteq \mathbb{R}^n$, $k \geq 1$ and c is a k-cube in V, we define

$$\int_c \omega := \int_{[0,1]^k} c^* \omega \qquad \omega \in \underset{A \subseteq \mathbb{R}^n}{\Omega^k(A)}$$



↗ k form over I^k

↙ pull back of c

$c^* w$

$c : [0,1]^k \longrightarrow \mathbb{R}^n$

$c^* : \Lambda^k(\mathbb{R}^n_{c(p)}) \rightarrow \Lambda^k(\mathbb{R}^k_p)$

Note : If ω is a 0-form, i.e ω is a function $\mathbb{R}^n \longrightarrow \mathbb{R}$ and if

$c : \{0\} \longrightarrow V \subseteq \mathbb{R}^n$ is a 0-cube

we define $\int_c \omega = \omega(c(0))$

$$\int_c w = \int_{[0,1]^0} c^* w = \omega(c(0))$$

Defn: The integral of ω over a k-chain

$$c = \sum_{i=1}^{m} a_i c_i$$

defined by $\int_c \omega = \sum_{i=1}^{m} a_i \int_{c_i} \omega$

Theorm : Let $c : [0,1]^k \longrightarrow \mathbb{R}^n$ be a k-cube. Let $S = c([0,1]^k)$
If $\omega = f\, dz^{i_1} \wedge \ldots \wedge dz^{i_k}$ is a k-form on $\mathbb{R}^n$ containg S
then

$$\int_c \omega = \int_{[0,1]^k} (f \circ c) \det \left(\frac{\partial(c^{i_1}, \ldots c^{i_k})}{\partial(x^1, \ldots x^k)} \right)$$

↪ This is a matrix

ω is k-form on $\mathbb{R}^n$

$c : [0,1]^k \longrightarrow \mathbb{R}^n$

$$\int_c \omega = \int_{[0,1]^k} c^* \omega = \int_{[0,1]^k} (f \circ c) \det(a_{p,q})$$

whose $a_{p,q} = \frac{\partial c^{i_p}}{\partial x^q}$

Eg: $\omega = P dx + Q dy + R dz$ is 1-form on $U \underset{\text{open}}{\subseteq} \mathbb{R}^3$ and

$\varphi : [0,1] \longrightarrow U$ is C^∞ function

(i.e $\varphi([0,1])$ is C^∞ curve in U)

$(C_\gamma = \varphi([0,1]))$

then

$$\int_{C_\gamma} P dx + Q dy + R dz = \int_{[0,1]} P(\gamma(t)) \frac{d\gamma^1}{dt} dt + \int_{[0,1]} Q(\gamma(t)) \frac{d\gamma^2}{dt} dt + \int_{[0,1]} R(\gamma(t)) \frac{d\gamma^3}{dt} dt$$

↙ For this case $\underset{\Uparrow}{\overset{f}{}}\ \omega = P dx$

$f \circ c = P(\gamma(t))$

and $a_{p,q} = \frac{\partial c^1}{dt} dt = \frac{\partial \gamma^1}{dt} dt$

Theorm : (Stoke's theorm) If ω is $(k-1)$ form on an open set $U \subseteq \mathbb{R}^n$ and
c is a k-chain in U, then

$$\overset{\text{k form}}{\int_{\underset{\text{k-chain} \hookrightarrow}{c}}} d\omega = \int_{\underset{\leftarrow \text{k-1 chain}}{\partial c}} \overset{\swarrow \text{k-1 form}}{\omega}$$

15th April :

Theorm : (Stroke's theorm) If ω is a $(k-1)$ form on an open set $U \subseteq \mathbb{R}^n$ and c is a k-chain in U, then

$$\int_c d\omega = \int_{\partial c} \omega$$

proof : case I : $c = I^k$ and ω is $(k-1)$ form of the type :

$$f\, dx^1 \wedge dx^2 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$$

notation means that $\widehat{dx^i}$ is omited

It is enough to prove the theorm for $k-1$ forms of this type

to show: $\int_{I^k} d(f\, dx^1 \wedge dx^2 \dots \wedge \widehat{dx^i} \wedge \dots dx^k) = \int_{\partial I^k} f\, dx^1 \wedge \dots \widehat{dx^i} \wedge \dots dx^k$

now on left

$$d(f\, dx^1 \wedge \dots \widehat{dx^i} \dots \wedge dx^k)$$
$$= D_i f\, dx^i \wedge dx^1 \wedge \dots \dots dx^k$$
$$= (-1)^{i-1} D_i f \; dx^1 \wedge dx^2 \wedge \dots dx^k$$

therefore :

$$\int_{I^k} d(f\, dx^1 \wedge \dots \widehat{dx^i} \wedge \dots dx^k)$$
$$= \int_{I^k} (-1)^{i-1} D_i f \; dx^1 \wedge \dots dx^k$$
$$= \int_{[0,1]^k} (-1)^{i-1} D_i f$$
$$= (-1)^{i-1} \int_{[0,1]^k} D_i f$$

Fubinis

$$\overset{\downarrow}{=} (-1)^{i-1} \int_0^1 \int_0^1 \dots \dots \left(\int_0^1 D_i f\, dx^i \right) dx^1 \dots \widehat{dx^i} \dots dx^k$$

FTC

$$\overset{\downarrow}{=} (-1)^{i-1} \int_0^1 \int_0^1 \dots [f(x^1, x^2 \dots 1, \dots x^k) - f(x^1, \dots 0, \dots x^k)] dx^1 \dots \widehat{dx^i} dx^k$$

$$= (-1)^{i-1} \int_{[0,1]^{k-1}} f(x^1, \dots 1, \dots x^k)$$
$$+ (-1)^i \int f(x^1 \dots 0, \dots x^k)$$

$$= (-1)^{i+1} \int_{[0,1]^{k-1}} f \circ I^k_{(i,1)} \; [0,1]^{k-1} + (-1)^i \int_{[0,1]^{k-1}} f \circ I^k_{(i,0)}$$

now right side of equation :

$$\partial I^k = \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} I_{(j,\alpha)}$$

$$\Rightarrow \int_{\partial I^k} f\, dx^1 \wedge \dots \widehat{dx^i} \dots \wedge dx^k$$

$$\int_{\partial I^k} f\,dx^1\wedge\dots\wedge\widehat{dx^i}\wedge\dots\wedge dx^k = \sum_{j=1}^{k}\sum_{\alpha=0,1}\int_{I^k_{(j,\alpha)}} f\,dx^1\wedge\dots\widehat{dx^i}\dots\wedge dx^k$$

(By defination of integration by chain)

$$= \sum_{j=1}^{k}\sum_{\alpha=0,1}(-1)^{j+\alpha}\int_{[0,1]^{k-1}}(f\circ I^k_{(j,\alpha)})\det\left[\frac{\partial(I^k_{(j,\alpha)})^{1,\dots,\hat{i},\dots,k}}{\partial x^1\dots x^{k-1}}\right]$$

$(k-1)\times(k-1)$ matrix of $\Delta I^k_{(j,\alpha)}$ obtained by selecting $1,\dots,\hat{i},\dots k$

leave out

we can show that
$$\det\left[\frac{\partial(I^k_{(j,\alpha)})^{1,\dots,\hat{i},\dots,k}}{\partial x^1\dots\quad x^{k-1}}\right] = \begin{cases} 0 \; ; & j\neq i \\ 1 \; ; & j=i \end{cases}$$

(this is trivial and from defination as if i not equal then row becomes 0)

so
$$\int_{\partial I^k} f\,dx^1\dots\widehat{dx^i}\wedge\dots\wedge dx^k = (-1)^{i+1}\int_{[0,1]^{k-1}} f\circ I^k_{(i,1)} + (-1)^i\int_{[0,1]^{k-1}} f\circ I^k_{(i,0)}$$

and so $\int_{\partial c}\omega = \int_c d\omega$ in this case

<u>Case 2</u>: If c is an arbitrary k-cube then

$$\int_{\partial c}\omega = \int_{\partial I^k} c^*\omega \quad \text{by defination}$$

and
$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega) \quad (\because \text{pull back commute})$$
$$= \int_{\partial I^k} c^*\omega \quad \text{by case I}$$
$$= \int_{\partial c}\omega \quad \text{by defination}$$

<u>Case 3</u>: If c is a k-chain:

$$c = \sum_{i=1}^{m} a_i c_i$$

then
$$\int_c d\omega = \sum a_i \int_{c_i} d\omega = \sum a_i \int_{\partial c_i}\omega \overset{\text{By case II}}{=} \int_{\partial(\sum a_i c_i)}\omega = \int_{\partial c}\omega$$

17th April:

Recap: $\int_{\partial c} \omega = \int_c d\omega$ (normal stokes theorem of multivariable)

classical notation of stokes theorem :

FTC: $\int_a^b F'(x)\,dx = F(b)-F(a)$

$c : [0,1] \longrightarrow \mathbb{R}$ by
$c(x) = a+(b-a)x$
$\downarrow$
1 cube ω is 0 form; the function
$\omega = F$
$\downarrow$
0-form $d\omega = F'(x)\,dx$

$\partial c = -c_{(1,0)} + c_{(1,1)}$

$= \underset{\substack{\downarrow \\ \text{0 cube}}}{`b'} - \underset{\substack{\searrow \\ \text{0-cube}}}{`a'}$

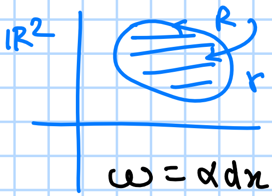
$\int_{\partial c} \omega = \int_{`b'} \omega - \int_{`a'} \omega = F(b) - F(a)$

$\underset{\text{0-chain}}{\swarrow} \quad \underset{\text{0-form}}{\downarrow}$

$\int_c d\omega = \int_a^b F'(x)\,dx$

$\int_a^b F'(x)\,dx = F(b) - F(a)$

Green's theorem :



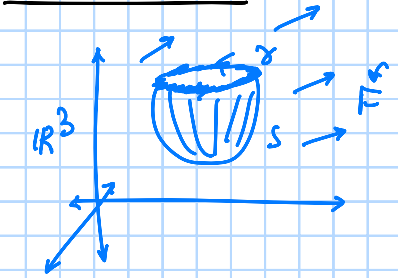
$\int_R \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right) dA = \int_\gamma \alpha\,dx + \beta\,dy$

$\omega = \alpha\,dx + \beta\,dy$
$d\omega = \frac{\partial \alpha}{\partial y}\, dy \wedge dx + \frac{\partial \beta}{\partial x}\, dx \wedge dy$

$= \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right) \underbrace{dx \wedge dy}$
dA can be shown in $\mathbb{R}^2$

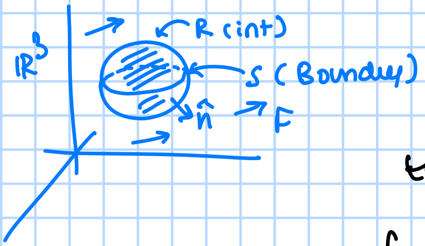
$\int_c d\omega = \int_{\partial c} \omega$
$c \downarrow R \qquad \partial c \downarrow r$

Stoke's theorm :



$$\int_S (\nabla \times F)\cdot \hat{n}\, dA = \int_\gamma F\cdot T\, ds$$

Divergence theorm :



$$\int_R \text{div}F\, dv = \int_S F\cdot\hat{n}\, dA$$

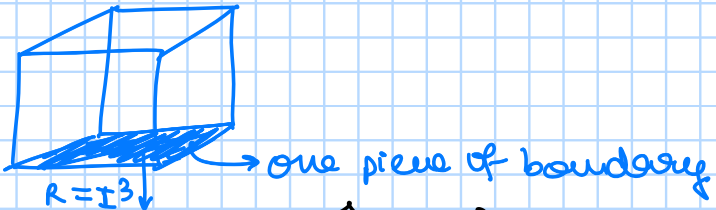
$$\omega = F^1 dy\wedge dz + F^2 dz\wedge dx + F^3 dx\wedge dy$$

then

$$d\omega = \text{div}F\, \underbrace{dx\wedge dy\wedge dz}_{dv \text{ (volume element)}}$$

$$\int_R d\omega = \int_{dR}\omega$$

$$\int_R \text{div}\, F.dv = \int_S F^1 dy\wedge dz + F^2 dz\wedge dx + F^3 dx\wedge dy$$



$\hat{n}\ (-e_3)$ $\quad F.\hat{n} = -F^3 \underbrace{dx\wedge dy}_{dA \text{ or area element}}$